Piecewise Deterministic Markov Process (PDMP)

As a modelling framework in RAMS

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Outline

• General introduction to PDMP
• PDMP in dynamic reliability
• PDMP for CBM – unit level
• Summary
PDMP - Intro

  - “Almost all continuous-time stochastic process models of applied probability consist of some combination of the following”:
    a) Diffusion
    b) Deterministic motion
    c) Random jumps

- Unified, highly developed theory.
- Ito Calculus, Stochastic differential equations
- Heterogeneous
- Special models and methodologies appropriate to specific problems
PDMP - Intro

M.H.A. Davis, 1984

“It can be argued that”:

1. The class of “piecewise-deterministic” Markov process, provides a general family of stochastic models covering virtually all non-diffusion applications
2. These process can be analyzed by methods which are directly analogous to those of diffusion theory.

Objective. To place non-diffusion models on the same footing as diffusion theory, with availability of:

• A “canonical model” including a variety of applications as special cases,
• General methods based on stochastic calculus for analyzing the canonical model

It is not implied that methods for studying non-diffusion models are obsolete. In many fields, efficient techniques for calculations have been built up, making use of the special structure of specific models.
PDMP - formalism

- Markov process consisting of a mixture of deterministic motion and random jumps
- Hybrid stochastic process \( \{I_t, X_t\}, t \geq 0 \) with values in a discrete-continuous space \( E \times R \)

\[
\{I_t, X_t\}
\]

Discrete component \hspace{3cm} Continuous component

**Law**

Determined by three local characteristics:

- Jump rate \( z(i, x) \)
- Flow \( \phi(i, x) \)
- Transition measure \( Q \)
PDMP - motion

The process starts at \((i, x)\):

1. Follows the flow \(\phi(i, x)\) until a first jump occurs at \(T_1\)
   A jump can occur:
   - Randomly, with rate \(z(i, x)\)
   - When the flow hits a boundary in the continuous-state space \(R\)

2. The post jump location is selected from the transition measure \(Q[(i, x), (j, y)]\)

3. The motion restarts from this point.

The flow is deterministic and generally described by differential equation
Dynamic reliability

- **Reliability**
  “the ability of an item to perform a required function, **under given** environmental and operational conditions and for a stated period of time” (Rausand M.)

- **Dynamic** reliability
  Extension of traditional reliability models and methods, with ones that are capable of **capturing the dynamics** of the operational and environmental conditions in which systems evolve.

Example

Case-study from

In the paper:

PDMP mentioned as framework for dynamic reliability (deterministic motion + random jumps)

However, the authors propose a modeling formalism based on Stochastic activity networks with flowsheet models, and a Discrete Event simulation algorithm
Dynamic reliability

• Problem

Air cooling system

• The air cooling system has an on-off control
• Works at a fixed working point $T_{cool}$

Change in room temperature

$$\frac{\partial T_{room}}{\partial t} = (Q_{in}) - (Q_{out} \cdot 1_{on}) + (Q_{CPU})$$

$$\frac{\partial T_{room}}{\partial t} = \frac{K}{J}(T_{ext} - T_{room}) - \frac{P}{J}(T_{room} - T_{cool})1_{on} + Q_{CPU}$$

• $T_{ext}$ – External temp
• $T_{room}$ – Room temp
• $T_{cool}$ – Cooling temp (fixed working point)
• $1_{on}$ – indicator function of the operative state of the cooling system (1–on, 0–off(standby))
• $K, P, J$ – Physical coefficients (heat transfer coefficients, heat capacity)
Dynamic reliability

\[
\frac{dT_{\text{room}}}{dt} = K \left( T_{\text{ext}} - T_{\text{room}} \right) - P \left( T_{\text{room}} - T_{\text{cool}} \right)_{\text{on}} + Q_{\text{CPU}}
\]

\[T_{\text{ext}}(t) = 20 + 15 \cdot \sin(7.17 \times 10^{-4}) + 5 \cdot \sin(0.2618 \cdot t)\]
\[T_{\text{room}}(0) = 20\ c\]
\[T_{\text{cool}} = 5\ c\]
\[Q_{\text{CPU}} = 0\]
\[K = 0.1\ W/C\]
\[J = 1\ W/C\]
\[P = 0.5\ W/C\]

Threshold\_off = 10 c
Threshold\_on = 15 c

“Failure free”
Dynamic reliability – random jumps

- **Weibull distributed lifetime**
  Failure rate, scale $\alpha$, shape $\beta$
  \[ z(t) = \frac{\beta}{\alpha} \cdot \left(\frac{t}{\alpha}\right)^{\beta-1} \]

- **Non-linear aging**
  \[ z(l) = \frac{\beta}{\alpha} \cdot \left(\frac{l}{\alpha}\right)^{\beta-1} \]
  - $L$ denotes actual time in operation (air cooling system in “on” position, governed by deterministic dynamics)
Dynamic reliability - PDMP

Jump rate
\[ z(1, (t\_room, l)) = \frac{\beta}{\alpha} \cdot \left(\frac{1}{\alpha}\right)^{\beta-1} \]
\[ z(2, (t\_room, l)) = \frac{\beta}{\alpha} \cdot \left(\frac{1}{\alpha}\right)^{\beta-1} \]

Flow
\[ \phi(1, (t\_room, l)) = \left(\frac{K}{J} (T\_ext - T\_room) - \frac{P}{J}(T\_room - T\_cool), 1 \right) \]
\[ \phi(2, (t\_room, l)) = \left(\frac{K}{J} (T\_ext - T\_room), 0 \right) \]

Transition measure
- Random jumps (not boundary hit)
  \[ Q[(1, (x), (3, x))] = 1 \]
  \[ Q[(2, (x), (3, x))] = 1 \]
- Boundary hit
  \[ q[(1, (10, l)), (2, (10, l))] = 1 \]
  \[ q[(2, (15, l)), (1, (15, l))] = 1 \]

Model state:
\[ \{i, (t\_room, l)\} \]
\[ i = \{1,2,3\} \]
\[ x = (t\_room, l) \]
Dynamic reliability - PDMP

• Quantification
  – Simulations
    • Random jumps + deterministic motion (flow)
      (Matlab + Simulink)
  – Numerical approach
    • Based on discretization of the continuous state space and time
    • Probability mass balance
PDMP - numerical approach

Renny Arismendi, Anne Barros, Antoine Grall,

*Piecewise deterministic Markov process for condition-based maintenance models — Application to critical infrastructures with discrete-state deterioration*,
Reliability Engineering & System Safety, Volume 212, 2021, 107540, ISSN 0951-8320,

**Numerical approach:** (Prob. mass balance: Markov property + total probability)

\[
\begin{align*}
\pi_{n+1}\delta(j, y) & \approx \sum_{i \in I} \pi_{n\delta}(i, x)[\lambda(i, x)Q(i, x, j)\delta] \\
& + \sum_{k=1}^{N-1} \sum_{h=1}^{N} \sum_{w=h+k}^{N} \pi_{n\delta}(h, w)[\lambda(h, w)Q(h, w, k)\delta][q(k, z, j)] \\
& + \sum_{k=1}^{N} \pi_{n\delta}(k, w)[1 - \lambda(k, w)\delta][q(k, z, j)] \\
& + \sum_{k=1}^{N} \sum_{h=1}^{N} \sum_{w=h+k}^{N} \pi_{n\delta}(h, w)[1 - \lambda(h, w)\delta][q(k, z, j)] \\
& + \sum_{y \in T(k, x, j)} \pi_{n\delta}(y)[1 - \lambda(j, x)\delta][q(k, z, j)]
\end{align*}
\]

(6)

Approximation of the Chapman-Kolmogorov equation (backward)
Interesting remarks

- Simulations
  - Computational cost, dynamics

- Numerical approach
  - Discretization: fixed vs variable-step

100 hours
Dynamic reliability - PDMP

3 cases

- **‘Hot’ model**
  - Same failure rate when on or standby

- **‘Warm’ model**
  - The failure rate when in standby is 80% of the failure rate when on

- **‘Cold’ model**
  - No failure while on standby

\[
T_{room} = 10
\]

\[
T_{room} = 15
\]

\[
Z(L) (\text{Hot})
\]

\[
Z(L) (\text{Warm})
\]

\[
0.8 Z(L) (\text{Warm})
\]

\[
0 (\text{Cold})
\]
PDMP for CBM

Interventions:
- Deterministic durations
- Non-continuous monitoring (the state of the item is only revealed by the operator at inspections)
- Delay before maintenance task

Deterioration:
- Time-dependent transition rates

In our case: not modelling a physical law with a differential equation, then why PDMP?

Figure 1: Condition-based maintenance model
PDMP for CBM

\{I_t, X_t\}

- Discrete-state deterioration

- Continuous component
  - not related to any physical phenomena, but used as an artifact to track time, either for:
    - Not constant random jump rate.
    - Introducing jumps at specified times.
PDMP for CBM – bridge management

Periodic Inspections:

Laws & rules

Condition is assigned according to the damage degree / severity:
1. Small
2. Medium
3. Large
4. Critical

Maintenance scheduling

Based on the condition assigned at inspection, laws and rules dictate when to maintain.

Maintenance action:
1. No action required
2. Action required between 4 to 10 years
3. Action required between 1 to 3 years
4. Action required before 6 months
Assumptions:
- Constant jump rate
- Perfect periodic inspections
- Delay (fixed duration) before maintenance
- Good-as-new repairs
PDMP for CBM

Model state \( \{I_t, X_t\} = \{(i_A, i_B), (x_A, x_B, t)\} \)

- \( i_A \) - deterioration state of the structure
- \( i_B \) - type of maintenance scheduled
- \( x_A \) - date of next inspection
- \( x_B \) - date of next maintenance action
- \( t \) - time

**Jumps at random times**

Used to model the deterioration process of the structure

- The deterioration state \( i_A \), jumps to a more deteriorated state
- The type of maintenance \( i_B \) does not change

**Jumps at deterministic times**

Used to model the inspection and maintenance of the structure

**Inspection, \( t = x_A \)**

Used to model the deterioration process of the structure

- The type of maintenance \( i_B \), is updated
- The date of next inspection \( x_A \), is updated
- A maintenance action \( x_B \) is scheduled or re-scheduled

**Maintenance, \( t = x_B \)**

Used to model the deterioration process of the structure

- The discrete component \( (i_A, i_B) \) jumps to \((1,1)\)
- The date of next inspection \( x_A \), does not change
- The date of next maintenance action \( x_B \) is set to infinite
PDMP for CBM

- Simulations

Model state \( \{I_t, X_t\} = \{(i_A, i_B, (x_A, x_B, t))\} \)
PDMP for CBM

- Expected cost per unit of time

\[ E[C] = E[N_{in}]C_{in} + E[N_{nr}]C_{nr} + E[N_{lr}]C_{lr} + E[N_{cr}]C_{cr} \]  \hspace{1cm} (13)

Illustration with symbolic values for cost
PDMP (time-dependent rates)

- Example:
  - Two states
  - Weibull dist. Lifetime
  - Fixed repair duration (corrective)

\[ \{I_t, X_t\} \]

\( i = \{1,0\} \) where: 1-working, 0-failed

\( x \): amount of time spent in the current discrete state \( i \) a time \( t \)

**Flow**

\[ \phi(i, x) = 1 \]

**Jump rate**

The failure rate from \( i = 1 \) to \( i = 0 \) is:

\[ z(1, x) = \frac{\alpha}{\mu} \left( \frac{x}{\mu} \right)^{(\alpha-1)} \]

**Continuous variable**

Bounded when \( i=0 \) due to repair duration,
(intervention jump to \( i=1, x=0 \))

If age-based PM, then \( x \) can be bounded when \( i=1 \), resetting \( x \) to zero.

**The challenge:**

Numerical approach implementation
Summary

PDMP

• Markov process consisting of a mixture of deterministic motion and random jumps

• Hybrid stochastic process \( \{ I_t, X_t \} , t \geq 0 \) with values in a discrete-continuous space \( E \times R \). Can be considered a canonical model including a variety of applications as special cases.

• A general class of non-diffusion stochastic models that provides a framework for studying optimization problems

• It is not implied that methods for studying non-diffusion models are obsolete. In many fields, efficient techniques for calculations have been built up, making use of the special structure of specific models.

• Two approaches are commonly used in the quantification of such models: Montecarlo simulation and Finite-volume scheme