*Research article*

# Danish preservice teachers' solving of linear equations: A challenge for teacher education

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**Abstract:** Solving linear equations, which requires a relational understanding of the equals sign, predicts later mathematical learning. The Danish national curriculum, despite expectations of equation-solving competence, privileges flexible approaches to mathematical problem solving above the learning of standard algorithms, goals reflected in the regulations governing teacher education. This paper presents an evaluation of the linear equations-related competence of 50 first year preservice teachers. Undertaken at the start of their programme, the assessment comprised four tasks, each designed to offer the possibility of shortcut solutions – solutions that involve fewer steps and are executed faster than standard algorithms. The first two included only integer coefficients, while the final two incorporated a mixture of integers, fractions or decimals. Success rates on the equations with integer coefficients were relatively high at 83% and 76% respectively, while the rates for those with mixed coefficients were significantly lower at 48% and 54% respectively. Procedurally, most solutions involved a standard algorithm (56%) that involved an expansion of brackets, followed by procedures based either on a 'swap the side swap the sign' principle, a 'do the same to both sides' principle, or both. Shortcut opportunities were acknowledged more frequently on the mixed coefficient equations than on the integer, highlighting the appeal of standard algorithms for solving seemingly familiar equations and the need for reflection on those that appeared unfamiliar. Failures were typically due to inadequate understandings of the distributive law, the arithmetic of signed numbers, transposition of terms, or division of two integers when the divisor was larger than the dividend or, if smaller, not a factor of it. Overall, the results highlight the extent to which Danish preservice teachers are ill-prepared for a programme that assumes a base level of competence.

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Keywords: Denmark, Mathematics teacher education, linear equations, shortcut strategies

# 1 Introduction

In the context of school mathematics, algebra generally (Dougherty et al., 2017) and linear equations particularly (Capraro & Joffrion, 2006) are viewed as gatekeepers to mathematics-related study in higher education (Reeder, 2017). Navigating this pathway is a ‘rite of passage’ (Clark & Lovric, 2008) for which many students are under-prepared (Thomas et al., 2015). Indeed, research has found many engineering undergraduates (Godfrey & Thomas, 2008) and teacher education students (Hansson & Grevholm, 2003) failing to understand the necessary characteristics of a linear equation.

Where national curricula for the compulsory school exist, few do not expect students to learn how to solve linear equations, although the timing of the topic’s introduction and the complexity of expected outcomes vary (Andrews, 2016). In Denmark, the context of this study, school education, both compulsory and post-compulsory, is structured by government-specified curricula that outline in broad and largely non-prescriptive ways the mathematics students are expected to learn. With respect to compulsory school, the curriculum emphasises strategy flexibility over standard algorithms. In the context of linear equations, by the end of grade six students should be able to translate numbers and arithmetic expressions into simple equations and know that the equals sign is not an instruction to calculate but an assertion of equality between both expressions. It adds that students should be able to solve such equations by means of both informal and formal methods. By the end of compulsory school students are expected not only to have knowledge of strategies for solving equations but be able to set up and solve equations and simple inequalities with and without digital tools (Ministry of Children and Education, 2019).

Students entering Danish preservice teacher education are obliged to study mathematics during their post-compulsory school years (Retsinformation, 2023). In particular, they are expected to be able to solve equations of the first and second degree by means of analytical, graphical and digital methods in both context-dependent and context-independent situations (Ministry of Children and Education, 2025). In light of such expectations, it would seem reasonable to investigate Danish preservice teachers’ linear equations-related competence.

## 1.1 The competences and didactics of equation solving

A core requirement for understanding and solving linear equations, as emphasised in the Danish curriculum, is an appropriate understanding of the equals sign. When taught arithmetic children frequently learn to see the equals sign as an instruction to operate (Kaput et al., 2007; Stephens et al., 2013). However, highlighting the problems created by such learning, equation solving requires a relational understanding of the equals sign as a statement of equality between two expressions (Alibali et al., 2007), a conceptualisation that some have argued, which we do not pursue further, can be subdivided into a relational-computational understanding and a relational-structural understanding (Stephens et al., 2013). Consequently, despite evidence that relational understanding typically develops by grades five or six (Rittle-Johnson et al., 2011), equation solving creates challenges for many students because operational perspectives on expressions like $3x + 1$ prevent their being construed as objects to be operated on (Kieran, 2004).

All the above underpins how students respond to equations-related instruction, and here we make a distinction between two forms of equation. There are arithmetical equations, which have the unknown in one expression only and can always be solved by operation reversal. They require little more than knowledge of arithmetic (Linchevski & Herscovics, 1996) and are, essentially, operational (Kieran, 1992). Then there are non-arithmetical or algebraic equations, which have unknowns in both expressions and cannot be solved by operation reversal. These require learners not only to "understand that the expressions on both sides of the equals sign are of the same nature" (Filloy & Rojano, 1989, p. 19) but also that they are able to operate on the unknown as an entity and not a number. They are essentially relational.

Historically, algebraic equations have been solved by one of two methods. The first, a 'swap the side swap the sign' (SSSS) procedure, is based on the transposition of terms from one side of the equation to the other. The second, conceptually underpinning a 'do the same to both sides' (DSBS) procedure, relies on operations undertaken on both sides of the equation simultaneously (Caglayan & Olive, 2010; Wasserman, 2014). Didactically, the former, typically teachers' preferred approach in a number of South-East Asian cultures (Cai et al., 2005; Ngu et al., 2023), has been criticised for masking mathematical understanding (Star & Seifert, 2006). The latter, observed in many Western cultures (Andrews & Sayers, 2012; Marschall & Andrews, 2015) and frequently exploiting the balance scale (Andrews, 2003; Stephens et al., 2022; Vlassis, 2002) has been criticised by cognitive load theorists for making demands in excess of those made by the former (See Boulton-Lewis et al., 1997; Ngu et al., 2015).

Mathematics textbooks typically privilege approaches underpinned by culturally-situated traditions (Larson & Larsson, 2024). In the context of the United States, textbooks and teachers tend to promote a *canonical* method for solving linear equations (Buchbinder et al., 2015, 2019), which comprises five steps, namely:

1. Use the distributive property to remove the grouping symbols.
2. Simplify the expressions on each side of the equals sign.
3. Use the addition and/or subtraction properties of equality to get the variables on one side of the equals sign and the numbers without variables on the other side of the equations sign.
4. Simplify the expressions on each side of the equals sign.
5. Use the multiplication or division property of equality to solve (Buchbinder et al., 2015, pp. 4/5).

Interestingly, the presentation of such rules does not preclude either SSSS or DSBS as an underlying principle, although Rittle-Johnson and Star (2007) implicitly privilege DSBS in their intervention in which students were invited to compare standard and non-standard approaches. However, didactical emphases on the canonical method, frequently expected of teachers (Bieda et al., 2015), militate against flexibility in equation solving (Star & Rittle-Johnson, 2008; Star & Seifert, 2006), prevent students from understanding that equations may yield identities or contradictions (Huntley et al., 2007), and result in alternative but correct solutions being evaluated as incorrect (Buchbinder et al., 2019). Finally, with an explicit focus on strategy flexibility, several studies have examined the ability of students to identify shortcut strategies – strategies that "are more efficient because they involve fewer steps and fewer computations; thus they may be executed faster and with fewer errors" (Rittle-Johnson & Star, 2007, p.562) - when solving linear equations. For example, Xu et al. (2017), drawing on the earlier work of Star and Seifert (2006), presented Chinese grade seven students a series of equations, each of which was constructed in ways that offered the possibility of a shortcut solution. They found that while “the vast majority of participants were able to correctly solve most of the items” (Xu et al., 2017, p.7), many fewer were able to recognise the shortcut possibility and even fewer used it as their first-choice approach. Similar results were found by Hästö et al. (2019), who drew on the methods and tools of Xu et al. (2017), in their evaluation of Finnish grade eleven students’ equations-related strategy use.

## 1.2 Equation solving and preservice teachers

Research on preservice teachers’ linear equation solving competence, the focus of this paper, has yielded a range of outcomes, highlighting the impact of culturally-situated didactical norms. A number of these studies have drawn on the task shown below, whereby participants are shown, with no additional text, the solution to an equation and asked to explain it to someone who had missed the lesson in which it had been taught. Such tasks allow for both disciplinary and instructional explanations (Leinhardt, 2001). The solution, presented over four lines and designed to confound expectations that the unknown should always finish on the left-hand side, was

$$x + 5 = 4x - 1;\;\; 5 = 3x - 1;\;\; 6 = 3x;\;\; 2 = x$$

Early manifestations of the task involved Greek and Greek-Cypriot preservice teachers. Most, irrespective of nationality, wrote about knowns and unknowns before offering an SSSS narrative. The key distinction was that Cypriot students’ accounts typically included an objective for the equation solving process, which the Greek students’ did not (Andrews & Xenofontos, 2017). However, the narratives of students from both countries indicated substantial weaknesses from both disciplinary and instructional perspectives (Xenofontos & Andrews, 2017). Exploiting the same methodology, Andrews (2020) analysed Swedish first year preservice teachers’ written responses to the same task shortly before they received any mathematics teaching. Most expressed clear objectives concerning the identification of the unknown. Procedurally a third of participants wrote an SSSS-based narrative, while twice that proportion wrote something attributable to DSBS. Further, two fifths of those who wrote initially about SSSS, switched to DSBS. In sum, the majority of Swedish students had a clear understanding of the purpose of equation solving and a procedure underpinned conceptually by DSBS. In similar vein, a similar Norwegian study found a strong reliance on SSSS for the additive elements of a solution and a tendency to invoke DSBS for the multiplicative (Larson, 2024). Also, those students who presented a conceptual objective – emphasising the goal of finding the value of the unknown – typically invoked DSBS for all aspects of the solution. Overall, while Norwegian preservice teachers’ use of SSSS may have been informed by DSBS, few seemed to see a need to articulate it. Collectively, these studies highlight cultural variation in how preservice teachers construe the same task. However, the extent to which preservice teachers are able to identify and exploit shortcut strategies is unknown. Consequently, this paper aims to address this gap in the literature by means of an evaluation of not only Danish preservice teachers’ linear equations-related competence but also their ability to recognise and exploit shortcut strategies.

# 2 The current study

Drawing on a definition of strategy “as a procedure or set of procedures for achieving a higher level goal or task” (Lemaire & Reder, 1999, p.365) and the equations-related work of Xu et al. (2017), this paper is framed by the question

> How, and in what ways, do Danish preservice teachers address linear equations that incorporate shortcut opportunities?

The data for this paper derive from a broader study of Danish preservice teachers' competence with multidigit arithmetic, fractions arithmetic and linear equations. Procedurally, we have adapted the data collection tool used in Xu et al.'s (2017) study which exploited 12 items, each designed to elicit shortcut strategies to investigate Chinese school students' linear equations-related strategy use. One of the tasks used in their test was $7(x - 2) = 3(x - 2) + 16$, which, following the canonical method, would be solved by eliminating the brackets before proceeding with a conventional DSBS or SSSS approach. However, the expectation was that students would recognise the equality of the bracketed terms and simplify the whole to obtain $4(x - 2) = 16$, before recognising the factor of four and the potential for further simplification.

In the study presented in this paper, rather than construct new variables based on aggregate scores that focus on statistical generalities, as with Xu et al. (2017) and Hästö et al. (2019), we wanted to examine in depth students' responses to differently structured equations. Consequently, in order to facilitate this depth analysis, we drew on the four tasks shown in Figure 1. Tasks 1, 2 and 4 were exactly the same as in Xu et al. (2017), while task 3 was adapted to create a slightly more challenging equation than the one originally developed for children. In this instance, the second term of the left hand expression of the original equation, $4x$, was replaced by $5x$ to force an engagement with improper fractions at the final line. Also, in this study we exploited only the first two phases of Xu et al.'s data collection protocol. That is, students solved each equation by means of whatever strategy they felt was appropriate, before solving the same equation in as many additional ways as they could.

1 $3(x - 1) = 27$

2 $2(x + 3) + 3(x + 3) = 20$

3 $4(x + 2.5) + 5x = 4(x + 2.5) + 8$

4 $\frac{3x+3}{3} + \frac{4x+4}{4} = 4$

***Figure 1.*** *The four tasks drawn from Xu et al. (2017)*

Participants were 50 first-year preservice teachers with mathematics as a major, who were presented the instrument during the first mathematics teaching session of their programme. Students were informed of the aims of the project and that results would be used solely for research purposes and not form any element of their teacher education assessment. In particular, they were informed that the aim was to investigate how they approached mathematical tasks. They were also informed that participation was optional and that there would be no consequences for withdrawing. No student chose to withdraw.

Procedurally, each solution to each equation was read and an initial narrative annotated on an electronic copy of the script. This annotation included reference to whether the solution was correct, incorrect, incomplete or unattempted, along with details of any processes inferable from the text. This process was repeated twice. With each pass the narrative for each task was refined and a consistent vocabulary developed. Table 1 presents the narrative set for one student. It shows seven solutions to the four tasks of which five were correct. It also shows that two shortcut possibilities were recognised, one of which led to a correct solution.

***Table 1.*** *The coded solution set for one student*

| Task 1 | Task 2 | Task 3 | Task 4 |
|---|---|---|---|
| 1 Correct. Correct expansion; SSSS algorithm.<br><br>2 Correct. Recognised shortcut. SSSS algorithm | 1 Incorrect. Correct expansion; SSSS algorithm; Incorrect transposition of constants; Incorrect equation solved correctly. | 1 Correct. Recognised shortcut; SSSS algorithm; Leaves solution as improper fraction.<br><br>2 Correct. Correct expansion; SSSS algorithm; Leaves improper fraction. | 1 Incorrect. Recognised shortcut; SSSS algorithm with incorrect transposition of constant; Incorrect equation solved correctly.<br><br>2 Correct. Common denominator; SSSS algorithm; Recognised earlier failure. |

# 3 Results

Table 2 presents a summary of the results for each task, from which several inferences can be drawn. The number of students failing to attempt a task increased from the first to the last, with only one student failing to attempt the first task and 12 failing to attempt the last. In related vein, the total number of attempts declined from the first to the last, from a high of 76 to a low of 52. In relation to the total number of attempts made, the proportions of correct solutions for tasks one and two were considerably higher than those for tasks three and four. The overall success rate – the proportion of correct solutions (169) in relation to the total number of attempts (255) - was 66%. That is, 34% of all attempts were incorrect. In terms of multiple solutions, 26 students (52%) offered more than one solution to the first task, with the figures for tasks 2, 3 and 4 being 15 (30%), 16 (32%) and 14 (28%) respectively. Students who offered multiple solutions were typically more successful than those who offered only one, having a mean success rate of 69% in comparison with 62%. However, looking only at first solutions, 49 students attempted task 1 with a success rate of 88%. For tasks 2, 3 and 4, the figures were 47 (74%), 44 (43%) and 38 (71%) respectively. Overall, 179 first solutions were offered, with 124 (70%) being successful. Alternatively, acknowledging that had all students attempted all tasks, there would have been 200 first solutions, the figure of 124 correct first solutions would have yielded a success rate of only 62%.

***Table 2.*** *Summary data for each task (no task attracted more than three solutions)*

| | Task 1 | | Task 2 | | Task 3 | | Task 4 | |
|---|---|---|---|---|---|---|---|---|
| Solutions per task | Number of students | Number correct | Number of students | Number correct | Number of students | Number correct | Number of students | Number correct |
| 0 | 1 | 0 | 2 | 0 | 6 | 0 | 12 | 0 |
| 1 | 23 | 20 | 33 | 28 | 28 | 7 | 24 | 12 |
| 2 | 25 | 41 | 12 | 20 | 15 | 18 | 14 | 16 |
| 3 | 1 | 2 | 3 | 2 | 1 | 3 | 0 | 0 |
| | Total attempts | Total correct | Total attempts | Total correct | Total attempts | Total correct | Total attempts | Total correct |
| Total | 76 | 63 | 66 | 50 | 61 | 28 | 52 | 28 |
| % correct | | 83 | | 76 | | 46 | | 54 |

The figures of Table 3 show the distribution of shortcut strategies over the four tasks. Overall, 41 first solutions were coded as employing a shortcut strategy, although this use was largely determined by task characteristics: tasks 1 and 2 yielded only 5 such solutions, while tasks 3 and 4 yielded 36. The distribution of shortcut strategies in subsequent solutions were more evenly distributed, with 8 found with tasks 1 and 2, and 10 with tasks 3 and 4. A chi square test run on the table's four partial sums (thus eliminating cells with a value less than five) confirmed ($\chi^2 = 5.81$, $p = 0.016$) that such a distribution was unlikely to be due to chance.

Beyond any expectations that students would have recognised shortcuts, the default starting point seemed to accord with canonical methods; students typically expanded brackets whenever possible. For example, of students' 255 attempts, 142 (56%) began with the expansion of brackets (125 correctly and 17 incorrectly). This was more typically followed by the use of SSSS (111) and DSBS (87). Also, highlighting a problem faced by many students, 27 (11%) failed attempts were due to an inability to divide one integer by another, while, on a more positive note, 42 (16%) solutions were annotated in ways that showed the solver's thinking process. In the following, we attend to each of the four tasks in turn.

***Table 3***. *Frequency of shortcut strategy use on the four tasks*

| | Shortcut in first solution | Total for each pair of tasks | Shortcut in later solutions | Total for each pair of tasks |
|---|---|---|---|---|
| Task 1 | 3 | | 7 | |
| Task 2 | 2 | 5 | 1 | 8 |
| Task 3 | 16 | | 6 | |
| Task 4 | 20 | 36 | 4 | 10 |
| Total | 41 | | 18 | |

## 3.1 Task 1

With respect to task 1 and students' recognition and use of shortcut strategies, three first and seven second solutions showed evidence of students acknowledging the simplification embedded in the shortcut. Figure 2 shows two such solutions from different students, 1.1 being one of the three first solutions and 1.2 one of the seven second. In both cases the student concerned has recognised the factor of three, with 1.1 presenting an implicit SSSS and 1.2 an explicit DSBS followed by an implicit SSSS. Both solutions, highlighting a degree of mathematical sophistication, also show an awareness of chains of implication.

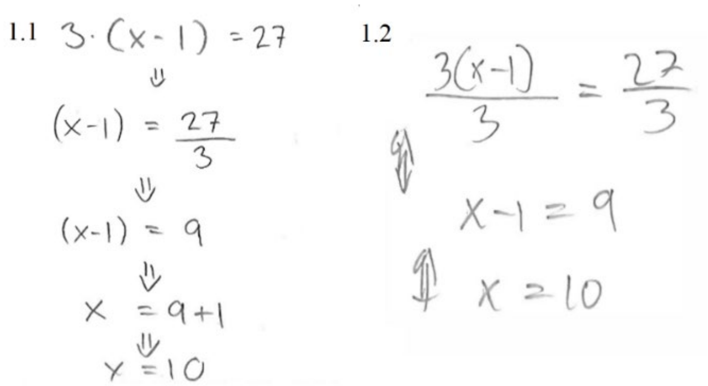


***Figure 2***. *Two shortcut solutions to task 1*

However, shortcut use was rare. The typical approach, undertaken on 54 of the 76 attempts, involved expanding the brackets before solving the simplified equation in various conventional ways, as shown in the four solutions of Figure 3. Solutions 1.3 and 1.5 narrate the expansion of the brackets, with 1.3 also referring to doing the same to both sides. Solutions 1.3 and 1.4 explicitly employ a DSBS process while solutions 1.5 and 1.6 implicitly employ SSSS, possibly as a process of operation reversal.

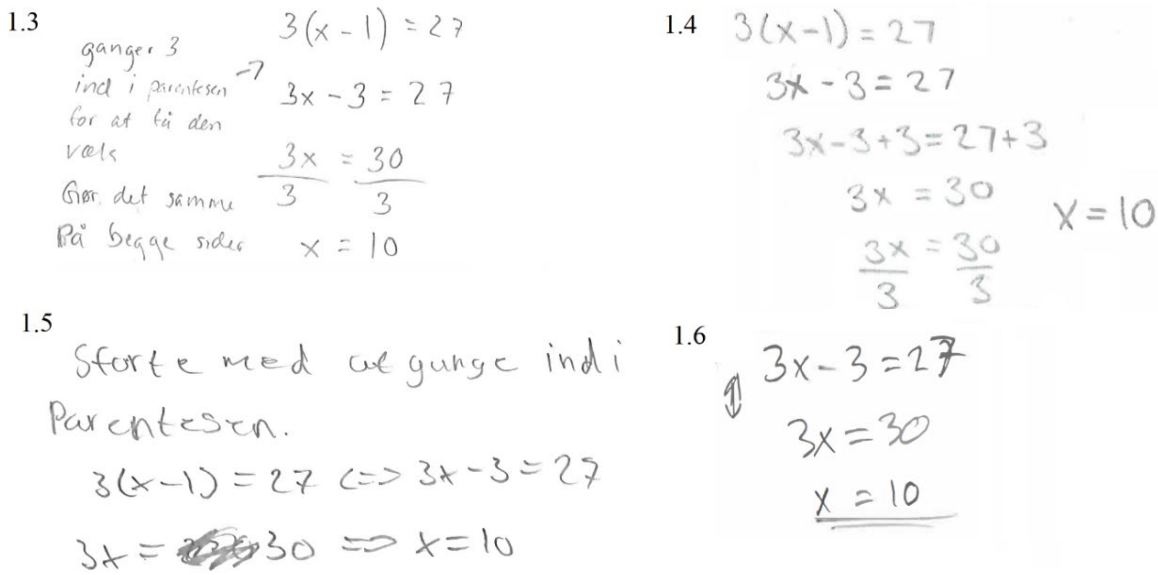


*Figure 3*. *Typical correct solutions involving expansion of brackets*

Eleven of the remaining correct solutions involved a simple numerical process, occasionally inspection. Figure 4 shows two such solutions. Solution 1.7 highlights the student's tacit awareness of the need to find a value for that will yield 9 when 1 is subtracted, while solution 1.8, having expanded the brackets, asks "what should 3 be multiplied by so that subtracting 3 yields 27, before answering, 10.

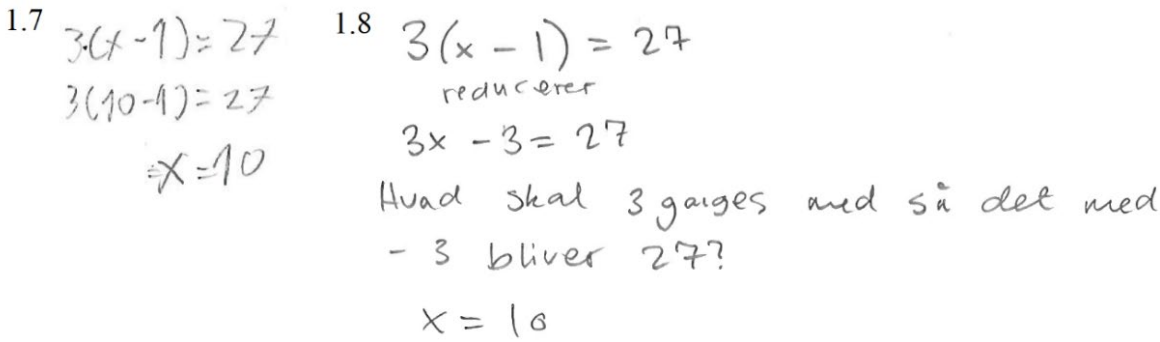


*Figure 4*. *Typical numerical solutions*

Overall, students addressed task 1 relatively successfully. The initial step of expanding brackets seemed well established, even to the extent that a handful of students annotated their solutions with text along the lines of "( ), · :, + -" to remind themselves of the rote-learnt order of operations. Incorrect solutions were typically a consequence of a failure to expand brackets correctly, with the most common error being related to misconceptions concerning when and under what circumstances operational signs change. Such errors can be seen in the two solutions shown in Figure 5, where both students have changed the sign of the constant when multiplying by the coefficient. In addition, solution 1.9 also conflates unknowns and constants to give $6x = 27$, which leaves the student uncertain as to what should be done, highlighting a problem we discuss below concerning non-integer solutions. Solution 1.10, albeit clumsily, manages to achieve a correct solution to the incorrectly derived equation.

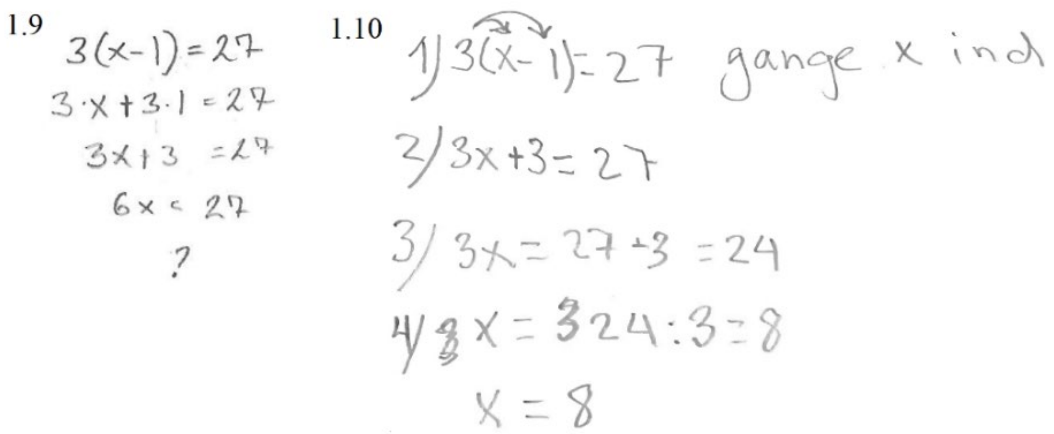


***Figure 5**. Typical bracket expansion errors*

## 3.2 Task 2

The figures of Table 2 show that two students failed to attempt the task. Of the remainder, 33 offered a single solution, 12 offered two and three offered three. In total, 66 solutions were offered, of which 50 (76%) were correct. With respect to shortcut strategies, only three solutions, two first and one second, showed evidence of students recognising the simplification embedded in the shortcut. Figure 6 shows all three of these solutions. Solution 2.1 clearly recognises the common term ($x$ + 3) and then the common factor (5) to derive a trivial equation. Solution 2.2, while drawing on a recognition of the common term, expands the brackets and then solves the remaining equation by means of an explicit DSBS. Finally, solution 2.3, after what might be described as a false start involving a complex transposition, seems to have had a flash of insight and identified the shortcut and solved a trivial equation.

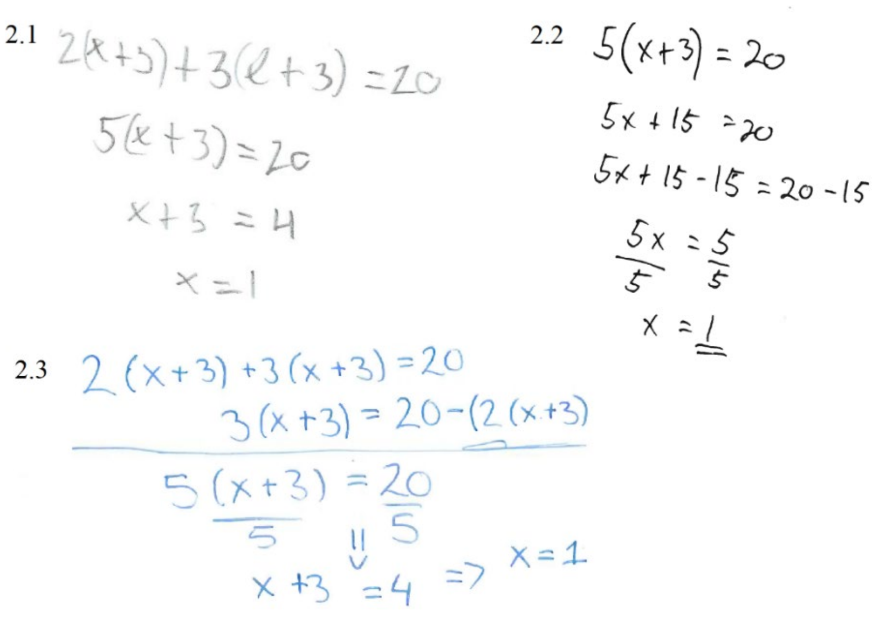


***Figure 6**. Three shortcut solutions to task 2*

As with task 1, the dominant approach to task 2 involved the expansion of brackets. Here, 39 correct solutions involved the expansion of brackets. Figure 7 shows three typical solutions of the type. The processes underlying solution 2.4 are difficult to discern, as the student concerned offered no indications as to what approach had been invoked. However, solution 2.5, by means of the insertion of both -6 and -9 on the right hand side, indicates clearly SSSS, while solution 2.6 is a clearly annotated DSBS supplemented by a check.

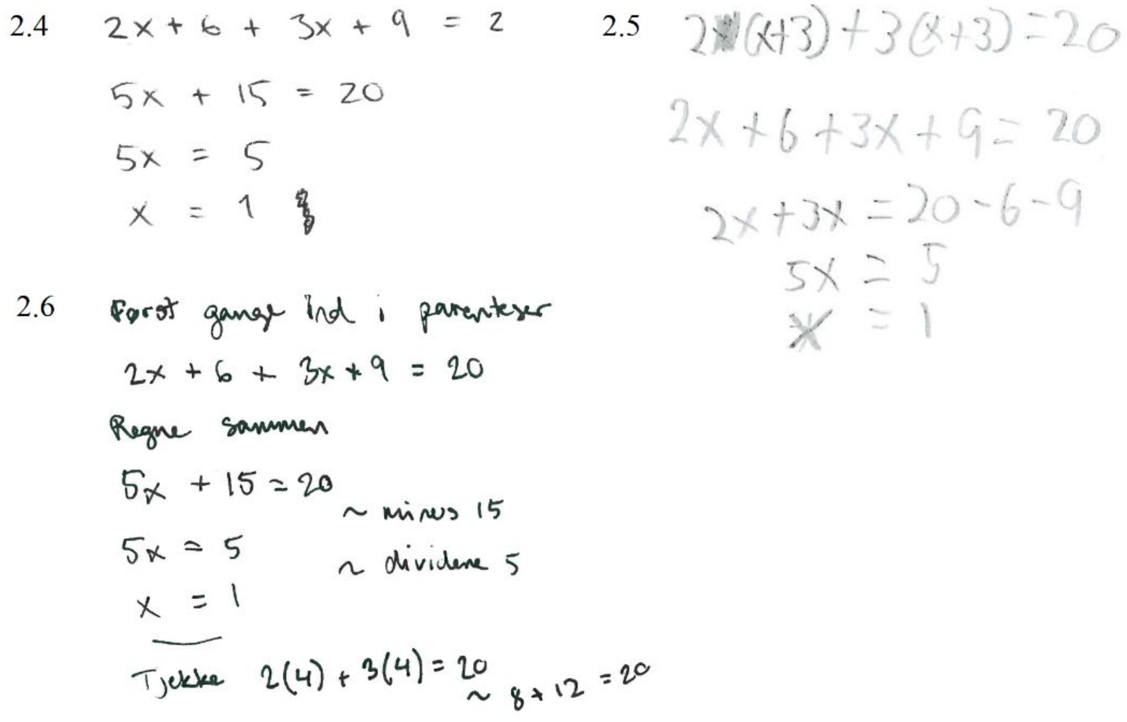


*__Figure 7__. Typical correct solutions to task 2 involving expansion of brackets*

Nine correct solutions to task 2 (a further two were unfinished) invoked elements of inspection. Figure 8 shows three such solutions, each of which offers different insights into the solution process. First, it is not unlikely that solution 2.7 was a consequence of a series of trial and error attempts. The assertion of 8 + 12 seems too confident to be an instant outcome of inspection. Solution 2.8 seems to have been a consequence, although it was not coded as such, of the student recognising the shortcut. Finally, solution 2.9, which was typical of the type, involved a correct expansion of the brackets before a numerical or inspection solution was employed. Of these three solutions, only 2.9 offered any indication of the student being explicitly aware of an algebraic approach to equation solving, although the sophistication of the numerical reasoning applied to 2.8 may ameliorate such concerns.

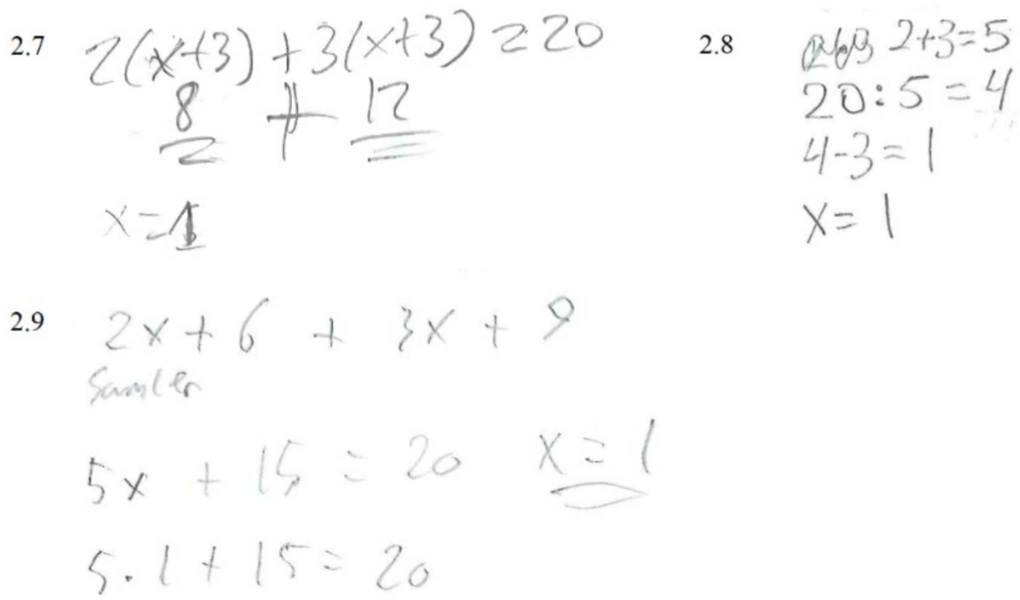


***Figure 8***. *Three numerical solutions to task 2*

In addition to the 39 correct bracket expansion-based solutions, there were seven failures, none of which could be described as a result of numerical error. Figure 9 shows three, each of which exhibits a common problem whereby the student concerned seems unable to complete a division when the dividend is not a multiple of the divisor. More importantly, while most of the expansion errors identified in relation to task 1 concerned the misapplication of vaguely remembered rules concerning the changing of operational signs, task 2 seemed to elicit a greater variety of misunderstandings. For example, solution 2.10, in multiplying the terms within the parentheses, indicates a misunderstanding of the distributive law. Solution 2.11, despite a valiant attempt to divide 76 by 5, exhibits at least two misconceptions concerning arithmetical operations: changing the signs of the constant term when expanding the brackets and multiplying (incorrectly completed) rather than adding the two negative constants. Finally, solution 2.12 shows a different misunderstanding of the distributive law, with only the unknown being subjected to multiplication and the constant left unchanged. These attempts, while showing clearly that the students concerned are familiar with linear equations and the processes by which they are solved, highlight a fundamental lack of basic algebraic and arithmetic competence and confidence.

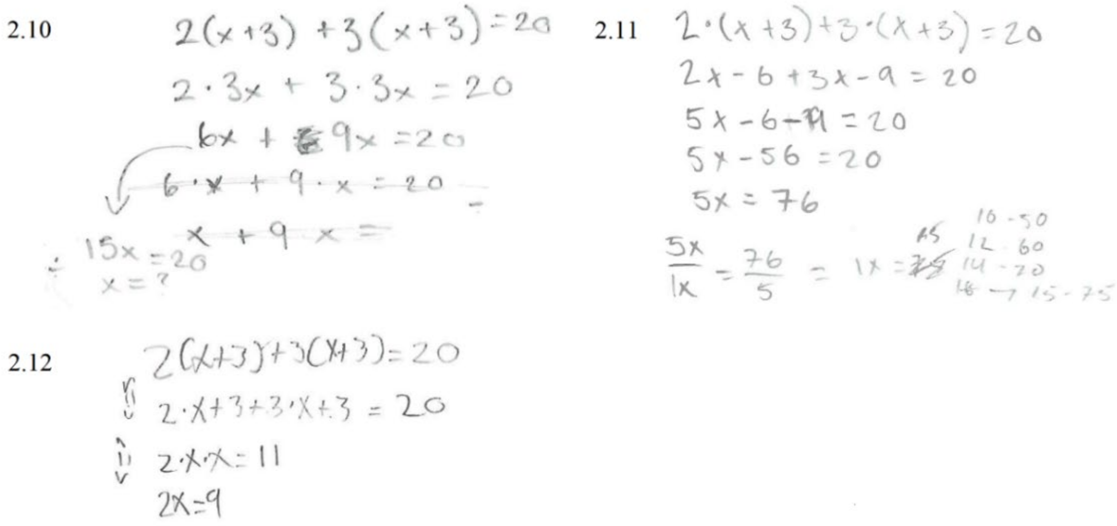

***Figure 9**. Three typical bracket expansion errors on task 2*

## 3.3 Task 3

The figures of Table 2 show that six students failed to attempt task 3. Of the remainder, 28 offered a single solution, 15 offered two and one offered three. In total, 61 solutions were offered, of which 28 (46%) were correct. With respect to shortcut strategies, 22 solutions (16 in students' first solutions) incorporated a recognition of a shortcut, although for eight of these the recognition was partial. Figure 10 shows four such solutions. With respect to solution 3.1, the student wrote, our translation, that "you have two of the same terms on both sides, so you can remove them". This was followed by the statement that $5x$ = 8, and the solution presented as an improper fraction. Such a solution, which incorporated little explicit algebra, highlights well the problem-solving efficiency of recognising a shortcut but also the fact, unacknowledged by most students, that an improper fraction has mathematical validity. In similar vein, solution 3.2, which begins with a reference to removing things, indicates not only an awareness of the shortcut but considerable resilience. Having commented that he or she was "wholly blank on 8 divided by 5", the student wrote that "five can be in 8 once with three remaining", followed by the assertion that one fifth is 0.2, and that three lots of 0.2 is 0.6 to give an answer of 1.6. Solution 3.3, resonant with 3.1, informs the reader that $4(x + 0.25)$ is on both sides of the equals sign and can be ignored, before applying DSBS. Finally, the student responsible for 3.4 simply asserts that $5x = 8$, before writing, as with 3.1, an improper fraction. All such solutions show a confident understanding of linear equations and the manner of their solution.

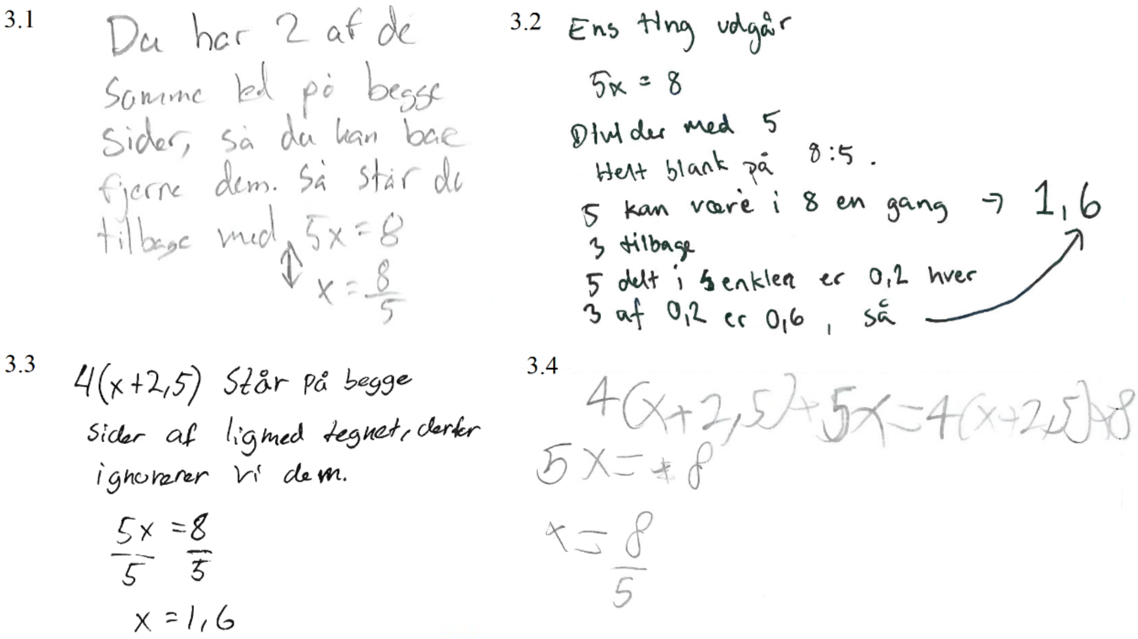


***Figure 10**. Four shortcut attempts to task 3*

Beyond the recognition of shortcut possibilities, the majority of approaches involved the expansion of brackets, some of which led to partial shortcuts and others to more conventional solutions, some correct and some not. Figure 11 shows four such solutions. Solutions 3.5 and 3.6 draw on correctly expanded brackets. However, while the annotations of 3.5 highlight a DSBS approach, nothing can be inferred from 3.6, although it is one of only a handful of solutions presented as a mixed number. Solution 3.7, following a correct bracket expansion, offers a partial shortcut whereby the same terms on both sides were simply crossed out to give a solution presented as both mixed number and decimal. Finally, solution 3.8, mirroring much of 3.7 highlights the ongoing problem for many students concerning the division of eight by five.

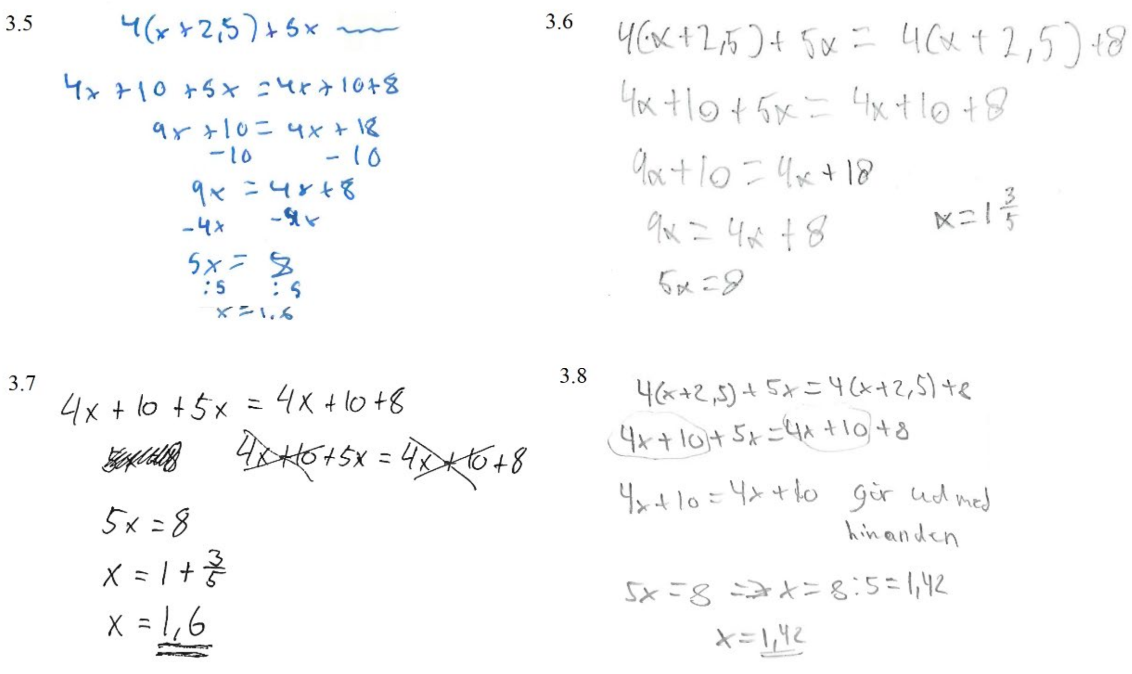


***Figure 11**. Four bracket expansion attempts on task 3*

For a variety of reasons, task 3 proved problematic for a large proportion of students. Many of these problems were a consequence of incorrect expansions of brackets, particularly the incorrect use of rote-learnt rules about the changing of signs during various operations. These have been discussed above and are not repeated here, as there is little new to be gained. However, a number of problems seemed to emerge from the existence of the unknowns on both sides. Figure 12 shows two out of seven such attempts, both of which draw on correctly executed expansion of brackets and subsequent collection of like terms. However, both, having arrived at the same simplified equation, are left incomplete. Solution 3.9 ends in confusion and 3.10 ends in an incorrect equation as a result of erroneously subtracting 8 from 9*x*.

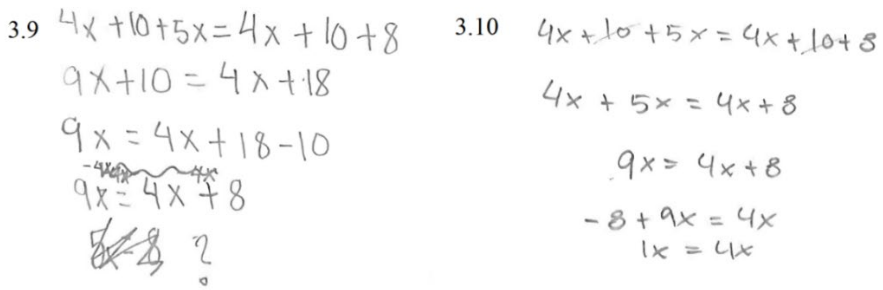

***Figure 12.*** *Attempts to task 3 showing confusion with the unknown on both sides*

## 3.4 Task 4

As with task 3, task 4 seems to have caused more problems for students than either tasks 1 or 2. The figures of Table 2 show that 13 students (26% of the cohort) failed even to attempt task 4. Of the remainder, 24 offered one solution and 13 offered two. In total, 50 solutions were offered, of which 28 (56%) were correct.

Of the 50 solutions offered, 20 first and 4 second solutions indicated some recognition of a shortcut possibility, which here was recognising that each of the two terms on the left-hand side of the equation was reducible to $x + 1$. Figure 13 shows two correct applications of the shortcut, which were representative of the 13, both first and second such solutions, that had proved successful. Their structure is remarkably similar and differ only in student choice as to what to annotate. In 4.1 the student has clearly demonstrated an awareness of the common factors and simply struck them out. In solution 4.2, the cancelling of the common factors was implicitly managed, followed by indications of self-correction and a clear privileging of a DSBS approach.

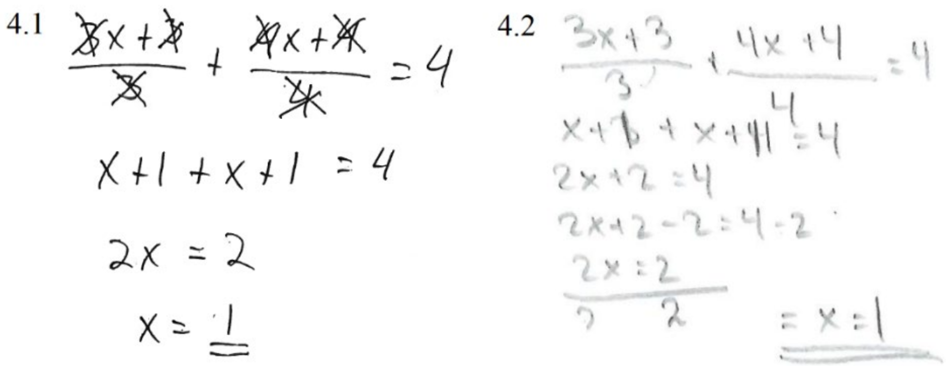

***Figure 13.*** *Two correct shortcut solutions to task 4*

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However, the success of shortcut solutions was considerably lower than on earlier tasks where shortcut possibilities had been recognised. Here, 11 of the 24 solutions in which a shortcut had been recognised and attempted led to failure. Figure 14 shows two typical incorrect attempts to exploit a shortcut on task 4. Solution 4.3, indicative of several similar solution attempts, shows a student unaware that the common factor applies to all terms, presumably not understanding the multiplicative significance of the coefficients of the unknowns. In addition, as found with the other tasks, the student concerned has clearly struggled to divide four by seven. Solution 4.4, by way of contrast, has recognised the need to eliminate the common factor from all terms but has, as with many earlier solutions to different tasks, misunderstood rules concerning the changing of signs under certain operations. Also, as with 4.3, the student concerned was unable to manage division when the dividend is not an integer multiple of the divisor.

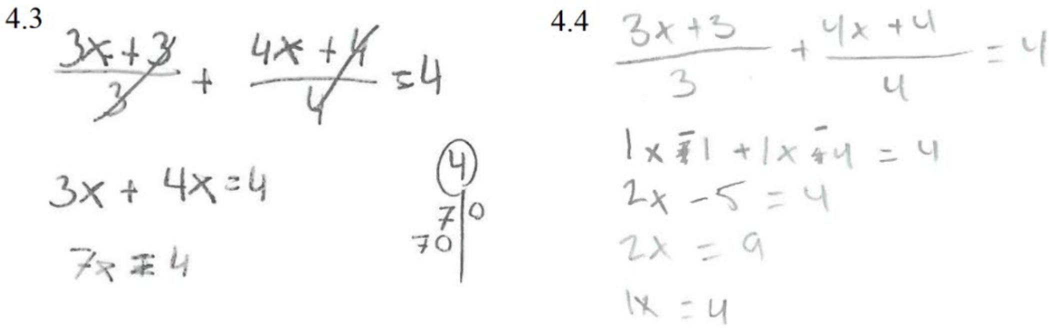


***Figure 14.*** *Two incorrect shortcut solutions to task 4*

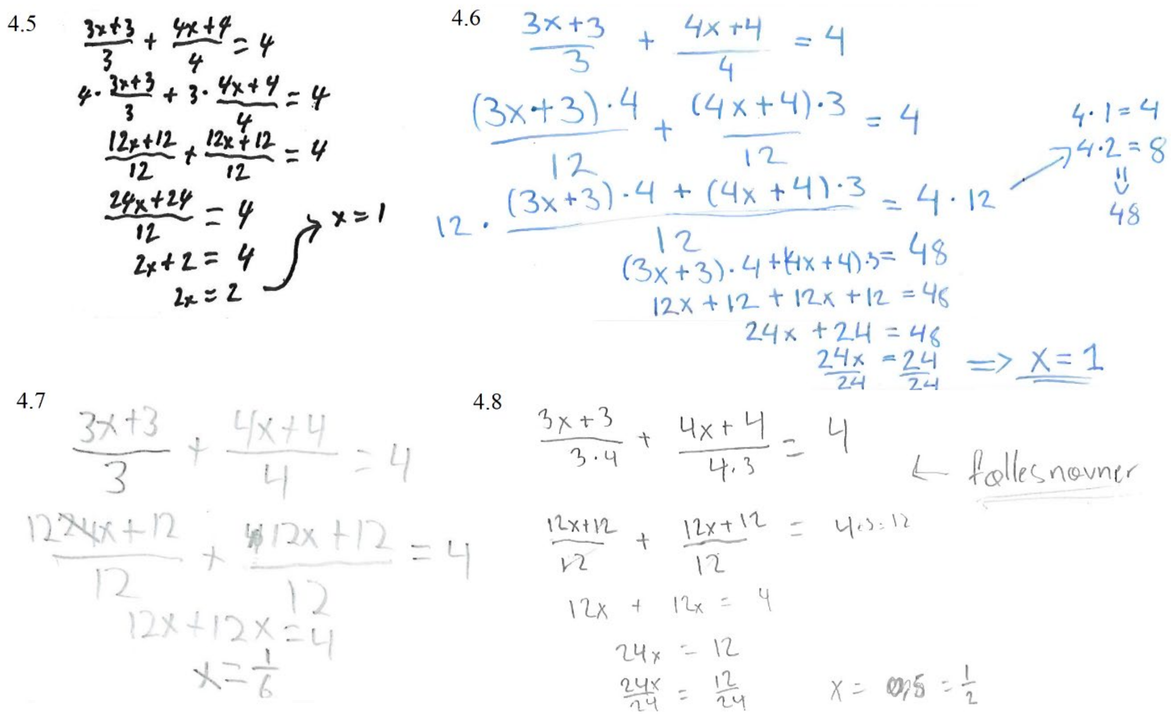


***Figure 15.*** *Correct and incorrect common denominator approaches to task 4*

Those solutions that did not invoke some sense of shortcut typically involved students exploiting their knowledge of fractions by calculating common denominators. Figure 15 shows two correct solutions, both of which involve lengthy and fairly dense calculations. The two are distinguished by whether attention is focused solely on the left-hand side of the equation, as with solution 4.5, or both sides, as with 4.6. Both are procedurally correct, but both are prone to the sorts of errors and misconceptions visible in 4.7 and 4.8. Attempt 4.7 has failed because the student has eliminated the constants from the two terms of the left-hand side of the equation. It is not clear why, although it is not unlikely that he or she has misapplied a rule concerning operations and changes of signs. That said and unlike many of his or her peers, this student seems confident in dividing 4 by 24. Solution 4.8, which is annotated by the term 'common denominator', shows considerable uncertainty as to how the common denominator should be implemented across the equation, leading to an incorrect conclusion, albeit a confident division of 12 by 24.

### 3.5 General comments

Interestingly, seven solutions were flukes, when multiple errors, typically concerning the distributive law or failures to calculate with signed numbers, cancelled each other out. For example, Figure 16 shows a solution to task 2 in which the student concerned has twice misapplied rules concerning the changing of signs. First, the student has assumed that when expanding brackets, multiplying a constant by a constant changes the sign. Second, summing two negative terms yields a positive. Ironically, what is now an accidentally correct equation has been completed correctly, including a clearly articulated DSBS division.

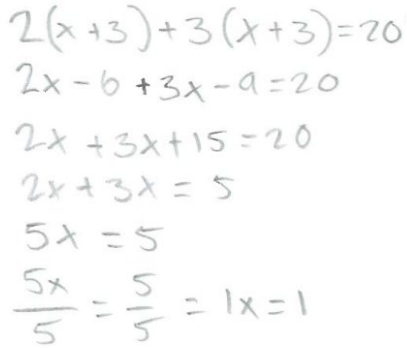


***Figure 16***. *A correct solution fluked by the correcting effects of two errors*

# 4 Discussion

The aim of this study was to evaluate not only Danish first year preservice teachers' linear equations-related competence but also their ability to recognise and exploit shortcut possibilities. In so doing, students were invited to provide multiple solutions for each of four tasks, each drawn directly from Xu et al. (2017) and designed to elicit shortcut strategies. Across these four tasks, each of which provided 50 opportunities for multiple solutions, students did so on 26, 15, 16 and 13 occasions respectively. In other words, on three of the four tasks, fewer than a third of all students offered multiple solutions, which compares poorly against the performance of Finnish high school students (Hästö et al., 2019).

In terms of shortcut recognition, the first two tasks, comprising only integer coefficients and constants, yielded only four solutions in which the shortcut was recognised. It is possible that these equations, being similar to those encountered at school, prompted a rote-learnt response. It is also possible that the shortcuts may have been recognised but ignored in the face of relatively simple applications of canonical methods. The latter two tasks, incorporating fractions and decimals, yielded 16 and 20 shortcut recognitions respectively. It is possible that these equations, which looked unfamiliar and less amenable to rote-learnt responses, may have prompted a closer examination. Were we to undertake this study again, we would invite a small sample of students to participate in clinical interviews (Hunting, 1997) in order to elicit more fully their responses to shortcut possibilities.

Levels of what is essentially a routine competence were surprisingly low. Admittedly, some students were very successful - one gave nine correct solutions to nine attempts - while others were not. For example, one student offered eight solutions, two for each task, and was successful on only one, which was a fortunate consequence of two errors negating each other. Of the 76 solutions offered by the 49 students who attempted task 1, 17% were incorrect. Of the 66 solutions offered by the 48 students who attempted task 2, 24% were incorrect. For task three, 46 students attempted the task, offering 61 solutions of which 55% were incorrect. Finally, 38 students attempted task 4, and of their 52 solutions, 46% were incorrect. The high failure rates on the two atypical tasks accord with an earlier study showing poor response rates on unfamiliar equations (Sfard & Linchevski, 1994). Overall, however, the 34% failure rate of these students was considerably higher than the 11.1% of Xu et al.'s (2017) grade 7 students.

Procedurally, when able to do so, students typically employed a canonical approach and expanded brackets (Buchbinder et al., 2015, 2019). While most were able to do this successfully, almost one in eight of these attempts failed due to the misapplication of rules concerning the distributive law or when operations change the sign of a number. Beyond the expansion of brackets, the use of both SSSS and DSBS was widespread, with many of the more successful students clearly using both. Also, there was evidence of a resonance with Larson's (2024) study, whose Norwegian preservice teachers tended to use SSSS on additive operations but DSBS on multiplicative, particularly division. However, the lack of annotations or other explicit indications meant that the underlying approaches to many solutions were impossible to infer. Overall, highlighting the significance of cultural context, the approaches adopted by the preservice teachers of this study seem more resonant with those of their Scandinavian (Andrews, 2020; Larson, 2024) than their Greek-speaking peers (Andrews & Xenofontos, 2017; Xenofontos & Andrews, 2017). That being said, the final line of many solutions highlighted an arithmetical problem concerning division when the divisor is numerically larger than the dividend or, if smaller, not a factor of the dividend.

However, there are few positives. Those students who invoked DSBS demonstrably understood the equals sign as a statement of equality between two expressions (Alibali et al., 2007) and, in so doing, showed a relational understanding of the equals sign (Capraro & Joffrion, 2006; Star & Seifert, 2006). These students, unlike some serving teachers (Asquith et al., 2007), should be well-positioned to help the children they will ultimately teach avoid acquiring an operational understanding of this key concept. This could not be said of those students who only invoked SSSS. The best that could be said of them is that their solutions offered no indications that they saw the equals sign as an instruction to operate (Kaput et al., 2007; Stephens et al., 2013). That said, few students offered anything indicative of a sophisticated substitutive understanding, whereby one expression is substituted with an equivalent expression (Donovan et al., 2022; Simsek et al., 2019). Moreover, acknowledging that participants were intending teachers, it is disappointing that only one in six solutions incorporated procedural annotations.

Finally, in a context in which ongoing reforms to the regulations governing teacher education have placed a stronger emphasis on content knowledge (Madsen & Jensen, 2023), the outcomes of this study foreground a fundamental problem for Danish teacher education, although the fact that data were collected at the beginning of these students' programme suggests there is time for remediation. The diverse failures identified above, whether of the equation solving process itself or of more fundamental processes concerning the principles of arithmetic, challenge the assumption that success on a national test is sufficient to guarantee Danish preservice teachers' mathematical knowledge for teaching, or the "mathematical knowledge needed to carry out the work of teaching mathematics" (Ball et al., 2008, p.395). Moreover, in a curricular context where strategy efficiency is privileged above standard algorithms, the dominance of the canonical method in students' solutions highlights a substantial lack of readiness for encouraging such ambitions in others. In other words, highlighting a need for substantial remediation, a lack of content knowledge, both conceptual and procedural, indicates that few of the preservice teachers of this study are equipped to determine, in respect of pedagogical content knowledge, "the most useful ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p.9). If there is encouragement to be found it is in participants' approaches to atypical tasks, where many recognised the shortcuts embedded in the tasks. However, why shortcuts were exploited on atypical and not typical tasks needs further investigation. Finally, while the tasks used proved highly effective in eliciting students' equation-related competence, they also proved to be powerful tools for diagnosing a range of arithmetical misconceptions largely unrelated to the solving of linear equations.

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