Fin 3005 V16

#1

a) ~~Hence~~ We see that $\frac{S_T}{S_0} = e^{(r-\frac{1}{2}\sigma^2)T+\sigma Z_T}$.

$$G_0 = e^{-rT} E_Q\left[\max\left(e^{(r-\frac{1}{2}\sigma^2)T+\sigma Z_T}, e^{gT}\right)\right]$$

$$= E_Q\left[e^{-\frac{1}{2}\sigma^2 T+\sigma Z_T}\cdot 1_A\right] + E_Q\left[e^{-rT+gT}\cdot 1_{\bar{A}}\right],$$

where $A = \left\{e^{(r-\frac{1}{2}\sigma^2)T+\sigma Z_T} > e^{gT}\right\}$ and

$\bar{A} = \left\{e^{(r-\frac{1}{2}\sigma^2)T+\sigma Z_T} < e^{gT}\right\}$.

We evaluate the last expectation first:

$$\pi_2 = E_Q\left[e^{(g-r)T} 1_{\bar{A}}\right] = e^{(g-r)T} Q(\bar{A})$$

$$= e^{(g-r)T} Q\left((r-\tfrac{1}{2}\sigma^2)T + \sigma \overbrace{Z_T}^{=\sqrt{T}\varepsilon,\ \varepsilon\sim N(0,1)} < gT\right)$$

$$= e^{(g-r)T} Q\left(\varepsilon < \underbrace{\frac{(g-r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{=-h}\right) = e^{(g-r)T} N(-h)$$

To evaluate the first expectation, we use the Radon-Nikodym derivative

$$\frac{d\hat{Q}}{dQ} = \frac{e^{-\frac{1}{2}\sigma^2 T+\sigma Z_T}}{\underbrace{E_Q\left[e^{-\frac{1}{2}\sigma^2 T+\sigma Z_T}\right]}_{=1}} :$$

$$E_Q\left[e^{-\frac{1}{2}\sigma^2 T+\sigma Z_T} 1_A\right] = E_{\hat{Q}}\left[1_A\right]$$

$$\cancel{= \hat{Q}(A) = \hat{Q}\left((r-\tfrac{1}{2}\sigma^2)+\sigma Z\right)}$$

By using Girsanov's theorem, we get that

$rT - \frac{1}{2}\sigma^2 T + \sigma Z_T$ under $\hat{Q}$ becomes

$$rT - \frac{1}{2}\sigma^2 T + \sigma\left(\sigma T + Z_T^{\hat{Q}}\right) = \frac{1}{2}\sigma^2 T + \sigma Z_T^{\hat{Q}}.$$

$$E_{\hat{Q}}\left[1_A\right] = \hat{Q}\Big(\overset{rT+}{\tfrac{1}{2}}\sigma^2 T + \sigma \overbrace{Z_T^{\hat{Q}}}^{=\sqrt{T}\varepsilon,\ \varepsilon \sim N(0,1)} > gT\Big)$$

$$= \hat{Q}\left(\varepsilon > \frac{(g - r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)$$

$$= 1 - \hat{Q}\Big(\varepsilon < \underbrace{\frac{(r - g + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}}_{\hat{h}}\Big)$$

$$= 1 - N(\hat{h}) = N(-\hat{h})$$

We then have that

$$G_0 = N(-\hat{h}) + e^{(g-r)T} N(-h).$$

b) $\frac{S_T}{S_0}$ is a gross return. To see this, buy $\frac{1}{S_0}$ units of the asset at a cost of $S_0 \cdot \frac{1}{S_0} = 1$. This investment gives you $\frac{1}{S_0} \cdot S_T = \frac{S_T}{S_0}$ at time T. The value of a gross return is 1. If $g \to -\infty$, $e^g \to 0$. Then $G_T \to \max\left(\frac{S_T}{S_0}, 0\right) = \frac{S_T}{S_0}$ is a gross return. Now, if the gross return has positive probability of becomming less than e^{gT}, the investor receives <u>more</u> than the gross return whenever

$\frac{S_T}{S_0} < e^{gT}$. Alternatively, we note that

$$\max\left(\frac{S_T}{S_0}, e^{gT}\right) = \frac{S_T}{S_0} + \underbrace{\max\left(e^{gT} - \frac{S_T}{S_0}, 0\right)}_{\text{a put option has positive value.}}$$

$\hookrightarrow$ G_T is a gross return (value $= 1$) + put (value > 0)

$\Longrightarrow G_0 > 1$.

c) $$H_0 = e^{-rT} E_Q\left[\prod_{i=1}^{T} \max\left(\frac{S_i}{S_{i-1}}, e^{g}\right)\right]. \quad (*)$$

For two RVs $\tilde{X}$ and $\tilde{Y}$, we know that

$$E[\tilde{X}\tilde{Y}] = E[\tilde{X}]E[\tilde{Y}] + \mathrm{cov}(\tilde{X}, \tilde{Y}).$$

We use the hint and can write $(*)$ as

$$H_0 = e^{-rT} \prod_{i=1}^{T} E_Q\left[\max\left(\frac{S_i}{S_{i-1}}, e^{g}\right)\right]$$

$$= \prod_{i=1}^{T} e^{-r} E_Q\left[\max\left(\frac{S_i}{S_{i-1}}, e^{g}\right)\right] = \prod_{i=1}^{T} \underbrace{E_Q\left[e^{-r}\max\left(\frac{S_i}{S_{i-1}}, e^{g}\right)\right]}_{(**)}$$

$(**)$ is just G_0 with $T=1$, $G_0(1)$.

Then

$$\underline{\underline{H_0 = \left(G_0(1)\right)^T}}$$

d) We note that if $\frac{S_i}{S_{i-1}} < e^g \ \forall i = 1, 2, \ldots, T$,

Then $H_T = \prod_{i=1}^{T} e^g = e^{gT}$.

But then $\frac{S_T}{S_0} = \prod_{i=1}^{T} \frac{S_i}{S_{i-1}} < e^{gT}$, so also

$$G_T = e^{gT}.$$

If $\frac{S_i}{S_{i-1}} > e^g \ \forall i$, then

$$H_T = \prod_{i=1}^{T} \frac{S_i}{S_{i-1}} = \frac{S_T}{S_0} > e^{gT},$$ so also

$$G_T = H_T.$$

If, for at least one i, $\frac{S_i}{S_{i-1}} < e^g$, it follows that $\prod_{i=1}^{T} \max\left(\frac{S_i}{S_{i-1}}, e^g\right) \geq \max\left(\frac{S_T}{S_0}, e^{gT}\right)$.

Thus
$H_T \geq G_T$ and then $H_0 \geq G_0$.

#2
- δ: rate of time preference. If you are impatient, you require a high r^f to postpone consumption

- $\gamma\mu$: μ is (expected) growth rate of consumption. High growth means we require high r^f to postpone consumption. A high γ means we like smooth consumption.

- $\frac{\gamma}{2}(\gamma+1)\sigma^2$: precautionary savings. We do not like risk, so we are willing to accept a lower r^f when we smooth/postpone our consumption. Higher consumption risk (σ^2), the lower r^f is.