Solutions FIN 3005 Asset pricing 2022

1. **Exercise 1**

a) Empirical phenomenon. Risk adjusted stock markets returns, e.g. measured by the Sharpe ratio are higher than reasonable risk aversion assumptions and consumption data should imply. The puzzle was established based on US post WW II data, but our textbook claims it holds for most countries. From economic theory one can derive the mathematical relationship

$$\frac{E[R^m] - R^F}{\sigma(R^m)} \leq \gamma\sigma(\Delta),$$

where the left hand side is the Sharpe-ratio of the market return, γ the relative risk aversion coefficient, and $\sigma(\Delta)$, the standard deviation of consumption. From US data (different sources report slightly different numbers, the above numbers are from our textbook) the risk premium $E[R^m]-R^F$ is 8%, market volatility $\sigma(R^m) = 16\%$, so that the Sharpe-ratio is 0.5. The standard deviation of consumption is 1% (again, different sources report different numbers, but they are not very different). These numbers imply that the coefficient should be 50. It is believed that reasonable values of γ are lower than 10, possibly around 2.

b) Something is wrong with the theory or the data. High values of γ ($\geq$ 50) lead to an interest rate puzzle; the risk free interest rate becomes extremely sensitive for small consumption shocks. Possible sources of errors: Extremely high return in all markets last 70 years (just a lucky outcome). Hard to measure true standard deviation of consumption, the standard deviation is typically based on annual data which therefore are smoothed over the year. The model is based on power utility, this function may be too simple to model preferences.

c) One set of resolutions to the puzzle consists of modelling more sophisticated preferences. Two examples are including *habits* (covered in class) and *recursive utility* (exercise set 3 - fall 22). Available consumption data are typically collected annually and therefore exhibit smoothing. Another set of solutions is to use more advanced consumption processes, possibly including jumps and/or stochastic volatility to obtain a higher value for the volatility of consumption.

Exercise 2 — 40 %

a)

$$\text{RRA} = -\frac{U''(C)}{U'(C)}C = -\frac{-\gamma C^{-\gamma-1}}{C^{-\gamma}}C = \gamma,$$

coeffisient of relative risk aversion.

b)

$$m_{t+1} = e^{-\delta}(\frac{C_{t+1}}{C_t})^{-\gamma} = e^{-\delta}e^{-\gamma\tilde{\Delta}} = e^{-\delta-\gamma\tilde{\Delta}}.$$

c) We can write

$$m_{t+1} = e^{-\delta-\gamma\tilde{\Delta}} = e^{\tilde{X}},$$

where $\tilde{X} \sim \mathcal{N}(-\delta - \gamma\mu, \gamma^2\sigma^2)$, m_{t+1} is, thus, *lognormally distributed.*

d)

$$R^f = \frac{1}{\mathbb{E}[m_{t+1}]} = \mathbb{E}[m_{t+1}]^{-1} = e^{\delta+\gamma\mu-\frac{1}{2}\gamma^2\sigma^2},$$

$$r^f = \ln(R^f) = \delta + \gamma\mu - \frac{1}{2}\gamma^2\sigma^2.$$

e) Assume that $\mu = \sigma = 0.01$, $\gamma = 2$, and $\delta = 0.02$.

$$r^f = 3.98\%.$$

f) Now, $\mu = 0, \sigma = 0.02$, $\gamma = 4$, and $\delta = 0.02$.

$$r^f = 1.68\%.$$

Exercise 3 — 45%

a) Using the the risk free rate and the market index, respectively, we know that

$$\pi_u + \pi_d = \frac{1}{R^f} = \frac{1}{1.04},$$

$$624\pi_u + 468\pi_d = 500.$$

It is easy to check that the values

$$\pi_u = \frac{25}{78}, \qquad \pi_d = \frac{25}{39},$$

solve these two equations.

b) $P_u = \max(507 - 624, 0) = 0$, $P_u = \max(507 - 468) = 39$.

$$P_0 = \pi_u P_u + \pi_d P_d = 25.$$

c) Claim π_u pays 1 in state *up* and 0 in state *down*. From $p = \mathbb{E}[mX]$,

$$\pi_u = p \cdot m_u \cdot 1 + (1-p) \cdot m_d \cdot 0 = p \cdot m_u.$$

Claim π_d pays 1 in state *down* and 0 in state *up*. From $p = \mathbb{E}[mX]$,

$$\pi_d = p \cdot m_u \cdot 0 + (1-p) \cdot m_d \cdot 1 = (1-p)m_d.$$

d)

$$\mathbb{E}[R_m] = p \cdot R_m^u + (1-p)R_m^d = \frac{5}{13}\frac{624}{500} + \frac{8}{13}\frac{460}{500} = 1.056.$$

e) From c)

$$m_u = \frac{\pi_u}{p} = \frac{5}{6},$$
$$m_d = \frac{\pi_d}{1-p} = \frac{25}{24}.$$

f)

$$P_0 = \mathbb{E}[mX] = p \cdot m_u \cdot P_u + (1-p) \cdot m_d \cdot P_d,$$

$P_u = 0$, and $P_d = 39$ from b). Thus,

$$P_0 = \mathbb{E}[mX] = 5/13 \cdot 5/6 \cdot 0 + 8/13 \cdot 25/24 \cdot 39 = 25.$$

g) We know form textbook/exercise set 3 (fall 22) that if the the stochastic disount function can be written as a linear function of the market return, the CAPM is equivalent to the state price/stochastic discount factor approach. I.e, if we can find constants a and b such that

$$a + bR_m^u = m_u,$$

and

$$a + bR_m^d = m_d,$$

then the CAPM produces the same prices as the other two approaches. But, the above two equations have two unknowns, so a solution (a, b) exists. (It is easy to show that the solution is $a = 5/3$ and $b = -625/936 = -0.667735$, but the solution itself is not important for this question).

Thus, we will get the the same price for the option if we price it by CAPM as we have got in b) and f), i.e. $\pi = P_0 = 25$.

h) $R_i^u = 0$ and $R_i^d = \frac{39}{\pi}$. We can, e.g., write $R_i = \frac{39}{\pi} 1\{S_1 = S_d\}$, where $1\{\cdot\}$ is the usual indicator function..

i)

$$\mathbb{E}[R_m^2] = p \cdot (R_m^u)^2 + (1-p)(R_m^d)^2 = 1.138176.$$

$$\text{Var}(R_m) = 1.138176 - 1.056^2 = 0.02304.$$

$$\mathbb{E}(R_m R_i) = p \cdot R_m^u R_i^u + (1-p) R_m^d R_i^d = \frac{22.464}{\pi}.$$

$$\text{Cov}(R_m, R_i) = \frac{22.464}{\pi} - 1.056 \frac{39}{\pi} 8/13 = \frac{-2.88}{\pi}.$$

j)

$$\beta = \frac{\text{Cov}(R_m, R_i)}{\text{Var}(R_m)} = -\frac{125}{\pi}.$$

k)

$$E[R_i] = R^f + \beta[\mathbb{E}[R_m] - R^f] = 1.04 - \frac{2}{\pi}.$$

l)

$$\mathbb{E}[R_i] = \frac{\mathbb{E}[X]}{\pi} = 1.04 - \frac{2}{\pi},$$

or, by inserting for $\mathbb{E}[X]$ and multiplying all terms by π,

$$\frac{8}{13} \cdot 39 = 1.04\pi - 2,$$

or

$$\pi = \frac{26}{1.04} = 25 = P_0,$$

as we knew from g).