Fin 3009 Fall 2023

1 a) With only one portfolio, it seems reasonable to rank the alternatives by their Sharpe ratio:

$$S_P = \frac{0.06}{0.15} = \underline{0.4}$$

$$S_Q = \frac{0.117}{0.26} = \underline{0.45}$$

$$S_M = \frac{0.08}{0.16} = \underline{0.5}$$ Pick portfolio M.

b) If the donation is invested in a well-diversified portfolio, which is the case here, the Treynor measure is suited to choose between P and Q:

$$T_P = \frac{0.06}{0.8} = \underline{0.075}$$

$$T_Q = \frac{0.117}{1.17} = \underline{0.1}$$ Pick portfolio Q.

c) In this case, you can rank the two portfolios – P and Q – by their information ratio:

$$IR_P = \frac{0.01}{0.05} = \underline{0.2}$$ Pick portfolio P.

$$IR_Q = \frac{0.027}{0.18} = \underline{0.15}$$

d) We need to find a portfolio of P and a risk-free asset with same σ as portfolio M:

$$w \cdot \sigma_P + (1-w)\sigma_{R_f} = \sigma_M \iff$$

$$w \cdot 0.15 + (1-w) \cdot 0 = 0.16 \iff$$

$$w = \frac{0.16}{0.15} = \frac{16}{15} = \underline{1\tfrac{1}{15}} \quad (=1.0\bar{6})$$

The excess return on this portfolio is

$$r_P^* = \frac{16}{15} \cdot 0.06 = \underline{0.064}$$

$$M^2 = 0.064 - 0.08 = \underline{-0.016}$$

↳ The portfolio gives less excess return than portfolio M, even if they have the same risk.

e) We have that

$$\bar{r}_P = \alpha_P + r_f + \beta_P \overbrace{\left[E r_M - r_f\right]}^{\bar{r}_M} - r_f = \alpha_P + \beta_P \bar{r}_M$$

$$S_P = \frac{\bar{r}_P}{\sigma_P} = \frac{\alpha_P + \beta_P \bar{r}_M}{\sigma_P} = \frac{\alpha_P}{\sigma_P} + \beta_P \frac{\bar{r}_M}{\sigma_P}$$

We have that

$$\beta_P = \frac{\mathrm{Cov}(P,M)}{\sigma_M^2} = \frac{\rho \sigma_P \sigma_M}{\sigma_M^2} = \frac{\rho \sigma_P}{\sigma_M}$$

By inserting for β_P in S_P, we get

$$S_P = \frac{\alpha_P}{\sigma_P} + \frac{\rho\sigma_P}{\sigma_M}\frac{\bar{r}_M}{\sigma_P}$$

$$= \frac{\alpha_P}{\sigma_P} + \rho\frac{\bar{r}_M}{\sigma_M} = \frac{\alpha_P}{\sigma_P} + \rho S_M.$$

For p to be the prefered portfolio, $S_P > S_M$:

$$\frac{\alpha_P}{\sigma_P} + \rho S_M > S_M \iff \frac{\alpha_P}{\sigma_P} > S_M(1-\rho).$$

Solving for ρ:

$$\rho > \frac{S_M - \frac{\alpha_P}{\sigma_P}}{S_M} = \frac{\frac{1}{2} - \frac{0.01}{0.15}}{\frac{1}{2}}$$

$$= \frac{\frac{15}{30} - \frac{2}{30}}{\frac{15}{30}} = \frac{\frac{13}{30}}{\frac{15}{30}} = \frac{13}{15} = \underline{\underline{0.8\overline{6}}}$$

$$\beta_P = \frac{\rho\sigma_P\sigma_M}{\sigma_M^2} = \frac{\frac{13}{15}\cdot 0.15 \cdot 0.16}{0.16^2} = \frac{13}{16} = \underline{0.8125}$$

Thus, $\underline{\underline{\beta_P > \frac{13}{16}}}$

(2) See Excel file "Problem_2.xlsx".

a) $RRA = -w \cdot \frac{u''(w)}{u'(w)} = -w\frac{-\gamma w_2^{-\gamma-1}}{w_2^{-\gamma}} = \underline{\underline{\gamma}}$

③ a) We are given

$$\Pi_T = \max(S_T^1 - S_T^2, 0)$$

The martingale approach gives

$$\frac{\Pi_0}{B_0} = E_t^Q\left[\frac{\Pi_T}{B_T}\right] = E_t^Q\left[\frac{\max(S_T^1 - S_T^2, 0)}{B_T}\right],$$

where $B_0 = 1$ and $B_T = e^{rT}$.

Let A be the event where $S_T^1 > S_T^2$ and let 1_A be an indicator function for the event A.

We can then write Π_T as

$$\max(S_T^1 - S_T^2, 0) = 1_A S_T^1 - 1_A S_T^2.$$

$$\Pi_0 = B_0 E_t^Q\left[1_A \frac{S_T^1}{B_T}\right] - B_0 E_t^Q\left[1_A \frac{S_T^2}{B_T}\right]$$

$$= S_0^1 E_t^Q\left[\underbrace{\frac{S_T^1/B_T}{S_0^1/B_0}}_{\text{R-N der.}} 1_A \frac{S_T^1}{S_T^1}\right] - S_0^2 E_t^Q\left[\underbrace{\frac{S_T^2/B_T}{S_0^2/B_0}}_{\text{R-N der}} 1_A \frac{S_T^2}{S_T^2}\right]$$

$$= S_0^1 E_t^{Q^{S^1}}[1_A] - S_0^2 E_t^{Q^{S^2}}[1_A]$$

$$= \underline{\underline{S_0^1 Q^{S^1}(A) - S_0^2 Q^{S^2}(A)}}$$

b)

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for(i=1; i ≤ Nsim, i++)
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$$S_T^1 = S_0^1 \cdot e^{(r-\frac{1}{2}\sigma^{1^2})T + \sigma^1\sqrt{T}\,\varepsilon^1}$$

$$S_T^2 = S_0^2 \cdot e^{(r-\frac{1}{2}\sigma^{2^2})T + \sigma^2\sqrt{T}\,\varepsilon^2}$$

$$\pi_0 = e^{-rT} \cdot \max(S_T^1 - S_T^2, 0)$$

$$sum = sum + \pi_0$$

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$$\pi_0 = \frac{sum}{Nsim}$$

c) It will reduce the value of the option. To see this, assume the returns are perfectly correlated and $\sigma_1 = \sigma_2$. Then the option can never expire in the money ($S_T^1 > S_T^2$) and the option is worthless.

With less than perfect correlation ($0 < \rho < 1$), the chance of getting a payoff becomes lower

↳ lower price.

4

The arbitrage-free forward price is given by

$$F_{t,T} = S_t \frac{1+r_{t,T}}{1+r^*_{t,T}} = 10 \cdot \frac{1{,}05}{1{,}07} = \underline{9{,}81}$$

There is an arbitrage opportunity. To exploit the arbitrage opportunity, you should "buy low and sell high". Here, you can synthetically buy the currency forward at 9.81 and sell it forward at 10.19.

Here is how it can be exploited:

1. Borrow 10 domestically + enter short forward to sell 1.07 foreign at time T
2. Exchange to foreign currency. $10 \cdot \frac{1}{S_t} = 10 \cdot \frac{1}{10} = 1$.
3. Invest 1 foreign currency at the risk-free rate in the foreign market: $1 \cdot 1{,}07 = 1.07$.
4. Sell 1.07 forward @ 10.19 = 10.90
5. Repay loan: $10 \cdot 1{,}05$ = 10.50

Arbitrage 0.40

Alternatively

1 Enter into forward contract to sell 1 unit of foreign currency for 10.19 at time T.

2 Borrow domestically, promise to repay 10.19 at time T. Gives a payout today of 9.705

3 Exchange 9.346 into foreign currency

4 Deposit $\frac{9.346}{10}$ @ 7% in foreign bank to get 1 foreign currency at time T.

This strategy gives a zero payoff at time T and $9.705 - 9.346 = \underline{0.359}$ at time t.

(The relationship between these two numbers ~~are~~ is:

$$\underbrace{\underbrace{0.359 \cdot 1.05}_{\text{Value at time T}} \cdot 1.07}_{\text{To get same number of contracts}} = 0.40$$

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