Exam in FY8304 Mathematical Approximation Methods in Physics
Thursday 9. December 2004
Time: 09:00—14:00

Exam aids: (Alternative B): Approved calculator.

This exam consists of 2 pages and a general addendum of 1 page.

Problem 1:
Consider the differential equation

\[ x^4 y''(x) + x^3 y'(x) - y(x) = 0. \]

a) Find and classify the singular points for this equation.

b) Determine the possible leading asymptotic behavior for \( y(x) \) when

i. \( x \to 0. \)

ii. \( x \to 1. \)

iii. \( x \to \infty. \)

Problem 2:
Determine leading behavior of the integral

\[ I(a) = \int_0^\infty dt e^{-t+a} t^a, \]

for

a) the limit \( a \to 0, \)

b) the limit \( a \to \infty. \)
Problem 3:
Consider the boundary value problem
\[
\frac{1}{100} y''(x) + (x^2 + 1)y'(x) - x^3 y(x) = 0, \quad y(0) = y(1) = 1. \tag{1}
\]
Introduce the perturbation parameter $\epsilon = 1/100$ in the first term, and use the boundary layer method to find

a) The outer and inner regions, and how the thickness of any boundary layer scales with $\epsilon$.

b) The outer and inner equations, and their solutions.

c) The condition for matching the outer and inner solution, and the uniform approximation.

d) Make a sketch of the uniform approximation to Equation (1).

Problem 4:
The WKB solution to the eigenvalue problem
\[
\epsilon^2 y''(x) + Q(x, E)y(x) = 0,
\]
where $Q(x) = V(x) - E$, has the leading order solution
\[
y(x) \approx \frac{1}{[Q(x, E)]^{1/4}} \exp \left( \pm \frac{1}{\epsilon} \int_x^1 dt \sqrt{Q(t)} \right).
\]

a) Show that the connection formula from region I (with $Q(x) > 0$ to region III (with $Q(x) < 0$) is
\[
y_{III} = 2C \frac{1}{[Q(x, E)]^{1/4}} \sin \left( \frac{1}{\epsilon} \int_x^{x_0} dt \sqrt{-Q(t) + \frac{\pi}{4}} \right)
\]
\[
\leftarrow y_{II} = C \frac{1}{[Q(x, E)]^{1/4}} \exp \left( -\frac{1}{\epsilon} \int_{x_0}^x dt \sqrt{Q(t)} \right).
\]
Assume that the two regions are connected by a region II, where $Q(x)$ is approximately linear.

b) Use the connection formula to find the WKB eigenvalue condition.

c) Consider bound states ($E < 0$) in the potential $V(x)$ given by
\[
V(x) = -\frac{V_0}{\cosh^2 x},
\]
and find the WKB approximation to the eigen-values $E_n$. Are there any limitations of the quantum number $n$?
Some information that may be useful

Airy functions

\[ \text{Ai}(x) \approx \frac{1}{2\sqrt{\pi}} x^{-1/4} \exp\left(-\frac{2}{3}x^{3/2}\right) \quad x \to \infty \]

\[ \text{Bi}(x) \approx \frac{1}{\sqrt{\pi}} x^{-1/4} \exp\left(\frac{2}{3}x^{3/2}\right) \quad x \to \infty \]

\[ \text{Ai}(x) \approx \frac{1}{\sqrt{\pi}} (-x)^{-1/4} \sin\left(\frac{2}{3}(-x)^{3/2} + \frac{\pi}{4}\right) \quad x \to -\infty \]

\[ \text{Bi}(x) \approx \frac{1}{\sqrt{\pi}} (-x)^{-1/4} \cos\left(\frac{2}{3}(-x)^{3/2} + \frac{\pi}{4}\right) \quad x \to -\infty \]

A numerical value

\[ \sqrt{\frac{2}{e}} = 0.85776... \]

A WKB integral

\[ \int_{-u_0}^{u_0} \sqrt{u_0^2 - u^2} \frac{du}{1 + u^2} = \pi \sqrt{u_0^2 + 1} \]