Eksamen TFY 4240: Elektromagnetisk teori
Thursday December 14 2006
kl. 09.00-13.00
English

Allowed help: C.
Rottmann: Matematisk Formelsamling (alle språkutgaver)
Barnett & Cronin: Mathematical Formulae
Øgrim: Størrelser og enheter i fysikken
Allowed calculator, empty memory, accordint to NTNU list
See also formulae page 7-10.

Each subsection has the same wight
Problems by:

Ola Hunderi       Jon Andreas Støvneng
Problem 1

In this problem we will study the multipole expansion of the static scalar potential $V(r)$. We assume known that at large distances the potential from a static dipole with dipole momentum $\vec{p}$, is given by:

$$V(\vec{r}) = \frac{\vec{p} \cdot \vec{r}}{4\pi \epsilon_0 r^3}$$

a) Four charges, one with charge $q$, one with charge $3q$ and 2 with charge $-2q$ are situated as shown in figure 1. All are at a distance $a$ from the origin. Find a simple approximate expression for the potential, valid at points far from the origin. Express your answer in spherical coordinates.

![Figure 1](image)

Given: $\vec{p} = \int \vec{r}' \rho(\vec{r}')d\tau'$

Next, calculate the potential at points far from the origin for four charges (two dipoles) placed along the $z$-axis:

- $+q$ is placed at $(0,0,-2a)$
- $-q$ is placed at $(0,0,-a)$
- $-q$ is placed at $(0,0,a)$
- $+q$ is placed at $(0,0,2a)$

Express your answer in polar coordinates. $a/r$ is small.

Hint: Expand the exact expression for the potential and look for terms that goes as $\frac{1}{r}$, $\frac{1}{r^2}$, $\frac{1}{r^3}$. Find in this way the monopole-, dipole- and quadrupole-contribution.
Or start with:

\[ V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 r} + \frac{\vec{p} \cdot \vec{r}}{4\pi\varepsilon_0 r^3} + \frac{1}{4\pi\varepsilon_0 r^3} \int r^{-2} \rho(\vec{r'}) \left( \frac{3}{2} \cos^2 \theta' - \frac{1}{2} \right) d\tau' + \ldots \]

c) Two line charges are lying in the xy-plane and parallel to the x-axis at a distance a from the axis. The left carries a charge -\lambda per meter and has y coordinate -a, the right carries a line charge equal to +\lambda per meter and has y-coordinate +a. Calculate the potential far away from the linecharges. Assume that a/r is small so you can expand.

Hint: Show first that the potential at a distance r from a single line charge is given by:

\[ V(r) = -\frac{\lambda}{2\pi\varepsilon_0} \ln(r) + \text{Const}. \]

**Problem 2**

a) Find the integral form of Faraday's law

\[ \oint \vec{E} d\vec{l} = -\frac{d}{dt} \int S \vec{B} d\vec{A} \]

and Gauss' law for the magnetic field

\[ \oint \vec{B} d\vec{A} = 0 \]

starting from the corresponding differential forms. Assume that the surface S does not change with time.

b) Find the general boundary conditions \( \Delta E_r = 0 \) and \( \Delta B_r = 0 \), i.e. that the tangential component of the electric and normal component of the magnetic field are continuous everywhere at a boundary.

We will now look at standing electromagnetic waves

\[ \vec{E}(x,y,z,t) = \vec{E}(x,y,z) \cdot e^{-i\omega t} ; \quad \vec{B}(x,y,z,t) = \vec{B}(x,y,z) \cdot e^{-i\omega t} \]
inside a rectangular cavity as seen in the figure. The walls are perfect conductors. The cavity has dimensions a, b, c as seen in figure 2.

![Figure 2](image)

**c)** The spatial part of $\bar{E}(x,y,z,t)$ is given by

$$\bar{E}(x,y,z) = A_x \cos k_x x \cdot \sin k_y y \cdot \sin k_z z \cdot \hat{x} +$$
$$A_y \sin k_x x \cdot \cos k_y y \cdot \sin k_z z \cdot \hat{y} +$$
$$A_z \sin k_x x \cdot \sin k_y y \cdot \cos k_z z \cdot \hat{z}$$

where $A_x$, $A_y$, and $A_z$ are unknown coefficients. Use the wave equation

$$\nabla^2 \bar{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \bar{E}}{\partial t^2},$$

and the boundary conditions to determine allowed values of the angular frequency $\omega$.

**d)** Determine the frequencies ($f=\omega/2\pi$) of the three lowest modes in a microwave oven with metallic walls and dimensions $a = 25$ cm, $b = 40$ cm and $d = 30$ cm. What is $\bar{B}(x,y,z)$ for the mode with lowest frequency?

---

**Problem 3**

We shall study dipole radiation from an oscillating charge and current distribution with time variation given by $e^{i\omega t}$. The vector potential in the general case is given by

$$\bar{A}(\bar{r},t) = \frac{\mu_0}{4\pi} \int \frac{\bar{J}(\bar{r}')}{|r_{ip}|} e^{i\omega t - \frac{\mu_0 |r_{ip}|}{c}} d\tau'$$
See the figure. In the equation is furthermore \( r_{ip} = |r - r'| = \pi \)

![Figure 3](image)

**Figure 3**

a) Explain why we get terms of the form \( e^{i(\omega t - kr)} \) in the integrand. Explain further which condition we assume when we in the radiation zone to lowest order write the vector potential in the form:

\[
\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{i(\omega t - kr)}}{r} \int \vec{J}(\vec{r}')d\tau'
\]

b) The vector potential in a) can be written as

\[
\vec{A}(\vec{r}, t) = i\omega \frac{\mu_0}{4\pi} p_0 \frac{e^{i(\omega t - kr)}}{r}
\]

Show that this vector potential leads to a B-field given by:

\[
\vec{B}(\vec{r}, t) = -\frac{\mu_0 p_0 \omega^2}{4\pi c} \sin \theta \frac{e^{i(\omega t - kr)}}{r} \hat{\phi} = \frac{\mu_0 \omega^2}{4\pi c} (\hat{\phi} \times \vec{p}_0) \frac{e^{i(\omega t - kr)}}{r}
\]

c) We shall next look at Poyntings vector. Show first that Poyntings vector in the radiation zone can be written in the form:

\[
\vec{S} = \frac{E_0}{\mu_0} E^2 \hat{r}
\]

Explain that the factor \( \frac{E_0}{\mu_0} \) has the dimension \( \Omega^{-1} \) and calculate its value.

One can show that the E-and B-field in the radiation zone are related by

\[
\vec{E} = -c (\hat{r} \times \vec{B}) \quad ; \quad \vec{B} = \frac{1}{c} (\hat{r} \times \vec{E})
\]

You are not asked to show this. Given the information you now have, show that Poyntings vector in the radiation zone can be written in the form:

\[
\vec{S} = Y \cdot (\hat{r} \times \vec{A})^2 \hat{r}
\]
Find Y.

Given: \( \vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \)

d) Assume now that we have a dipole of finite length, a dipolar antenna. See figure. Explain which of the assumptions in a) which are not fulfilled in this case. The vector potential for such an antenna is in general given by

\[
\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{i(\omega t - kr)}}{r} \int \vec{J}(\vec{r}') e^{ik\vec{r}' \cdot \hat{r}} d\tau'
\]

Assume that the dipole is a long thin antenna (see figure 4) oriented along the z-axis and where the current is given by

\[
I(z) = I_0 \cos kz \text{ for } \frac{-\lambda}{4} \leq z \leq \frac{\lambda}{4} \text{ and } 0 \text{ otherwise}. \quad \lambda = \frac{2\pi}{k}
\]

Show that the vector potential of the antenna in the radiation zone is given by

\[
\vec{A}(\vec{r}, t) = \frac{\mu_0}{2k\pi} I_0 \frac{e^{i(\omega t - kr)}}{r} \frac{\cos \left[ \frac{\pi}{2} \cos \theta \right]}{\sin^2 \theta} \hat{z}
\]

and calculate the angular variation of the radiated intensity \( \frac{dP}{d\Omega} \).

Given: \( \cos \alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2} \); \( \sin \alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i} \)