Motivation.
We calculated the contribution of a scalar field with mass $m$ to the vacuum energy density $\rho$ twice, once introducing a cutoff $M$ in momentum space and once using dimensional regularisation (DR). The two regularisation methods gave different results: The latter predicted $\rho \propto m^4$, while the former predicted $\rho \propto M^4$. Aim of the home exam is to obtain a better understanding why this happens and to decide which result is correct.

   a.) Determine the energy-momentum stress tensor $T^{\mu\nu}$ of the free, real scalar field with Lagrange density
   \[ L = \frac{1}{2} \eta_{\mu\nu} (\partial^\mu \phi) (\partial^\nu \phi) - \frac{1}{2} m^2 \phi^2 - \rho_0 \]
   and show that the vacuum energy density $\rho_0$ acts like a cosmological constant,
   \[ T^{\mu\nu} = \eta^{\mu\nu} \rho_\Lambda. \]
   b.) Find the energy-momentum stress tensor $T^{\mu\nu}$ for a perfect fluid in its rest frame (you can use the literature); compare the two stress tensors and show that the vacuum energy density $\rho_0$ and the cosmological constant have the equation of state (E.o.S.) $w = P/\rho = -1$ where $P$ denotes the pressure.

2. Momentum cutoff.
   Consider again a scalar field with mass $m$ and keep the mass dependence throughout.
   a.) Repeat the calculation leading to $\langle \rho \rangle \propto M^4$ [lecture notes (2.56-59)].
   b.) Calculate in the same way the contribution of zero-point fluctuations to the pressure $\langle P \rangle$ of the vacuum. [If you are unfamiliar with the definition of pressure in kinetic theory, then you can deduce the required connection by comparing the expressions (2,3) with the one you used in a.).]
   c.) Check if the E.o.E. $w = \langle P \rangle/\langle \rho \rangle = -1$ for a cosmological constant is satisfied for finite values of the cutoff $M$, i) for the leading $M^4$ terms, ii) for the subleading $m^4$ terms.

3. Dimensional regularisation.
   Redo the calculations of 2.) applying now DR, i.e. calculate
   \[ \langle \rho \rangle = \frac{\mu^{d-d}}{(2\pi)^{(d-1)}} \int d^{d-1}k \frac{\omega_k}{2} \]
   and $\langle P \rangle$ for $d = 4 - \varepsilon$. (Find the $d$ dimensional surface integral and express the remaining integral as Beta function.) Check again the E.o.S. $w$. Separate the expression for $\langle \rho \rangle$ into a finite and a pole part.

4. Interpretation.
   What is your interpretation of the results obtained (less than 100 words)?
Useful relations.
The number density $n$, energy density $\rho$ and pressure $P$ of a species $X$ follows as

\begin{align*}
  n &= \frac{g}{(2\pi)^3} \int d^3 p \ f(p) \quad (1) \\
  \rho &= \frac{g}{(2\pi)^3} \int d^3 p \ E f(p) \quad (2) \\
  P &= \frac{g}{(2\pi)^3} \int d^3 p \ \frac{p^2}{3E} f(p) \quad (3)
\end{align*}

where the factor $g$ takes into account the internal degrees of freedom like spin or colour.

The surface $\Omega_d$ of a unit sphere in $d$ dimensions is given by

\[ \Omega_d = 2\pi^{d/2}/\Gamma(d/2). \]

Euler’s beta function is defined by

\[ B(a, b) \equiv \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^\infty dt \ \frac{t^{a-1}}{(1+t)^{a+b}}. \quad (4) \]