1. Sphere $S^2$. 

The line-element of the two-dimensional unit sphere $S^2$ is given by

$$ds^2 = d\vartheta^2 + \sin^2 \vartheta d\phi^2.$$ 

a. Write out the geodesic equations and deduce the Christoffel symbols $\Gamma^a_{bc}$. (6 pts)

b. Calculate the Ricci tensor $R_{ab}$ and the scalar curvature $R$. (Hint: Use the symmetry properties of this space.) (6 pts)

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a. We use as Lagrange function $L$ the kinetic energy $T$. From $L = g_{ab} \dot{x}^a \dot{x}^b = \dot{\vartheta}^2 + \sin^2 \vartheta \dot{\phi}^2$ we find

$$\frac{\partial L}{\partial \phi} = 0,$$

$$\frac{\partial L}{\partial \vartheta} = 2 \cos \vartheta \sin \vartheta \dot{\phi}^2,$$

and thus the Lagrange equations are

$$\ddot{\vartheta} = 0 \quad \text{and} \quad \ddot{\phi} - \cos \vartheta \sin \vartheta \dot{\vartheta} = 0.$$ 

Comparing with the given geodesic equation, we read off the non-vanishing Christoffel symbols as $\Gamma^\vartheta_{\vartheta \phi} = \cot \vartheta$ and $\Gamma^\phi_{\vartheta \vartheta} = -\cos \vartheta \sin \vartheta$. (Remember that $2 \cot \vartheta = \Gamma^\vartheta_{\vartheta \phi} + \Gamma^\vartheta_{\phi \vartheta}$.)

b. The Ricci tensor of a maximally symmetric spaces satisfies $R_{ab} = Kg_{ab}$. Since the metric is diagonal, the non-diagonal elements of the Ricci tensor are zero too, $R_{\phi \vartheta} = R_{\vartheta \phi} = 0$. We calculate with

$$R_{ab} = g^{cd} R_{c ab} = \partial_c \Gamma^c_{ab} - \partial_b \Gamma^c_{ac} + \Gamma^c_{ab} \Gamma^d_{cd} - \Gamma^d_{bc} \Gamma^c_{ad}$$

the $\vartheta \vartheta$ component,

$$R_{\vartheta \vartheta} = 0 - \partial_\vartheta (\Gamma^\phi_{\vartheta \phi} + \Gamma^\vartheta_{\phi \vartheta}) = 0 - \Gamma^d_{bc} \Gamma^c_{\vartheta d} = 0 + \partial_{\vartheta} \cot \vartheta - \Gamma^\phi_{\vartheta \phi} \Gamma^\vartheta_{\vartheta \vartheta} = 0.$$ 

From $R_{ab} = Kg_{ab}$, we find $R_{\vartheta \vartheta} = Kg_{\vartheta \vartheta}$ and thus $K = 1$. Hence $R_{\phi \phi} = g_{\phi \phi} = \sin^2 \vartheta$.

The scalar curvature is (diagonal metric with $g_{\phi \phi} = 1/\sin^2 \vartheta$ and $g^{\vartheta \vartheta} = 1$)

$$R = g^{ab} R_{ab} = g^{\phi \phi} R_{\phi \phi} + g^{\vartheta \vartheta} R_{\vartheta \vartheta} = \frac{1}{\sin^2 \vartheta} \sin^2 \vartheta + 1 \times 1 = 2.$$ 

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[If you wonder that \( R = 2 \), not 1: in \( d = 2 \), the Gaussian curvature \( K \) is connected to the “general” scalar curvature \( R \) via \( K = R/2 \). Thus \( K = \pm 1 \) means \( R = \pm 2 \) for spaces of constant unit curvature radius, \( S^2 \) and \( H^2 \).]

2. Black holes.

The metric outside a spherically symmetric mass distribution with mass \( M \) is given in Schwarzschild coordinates by

\[
\text{ds}^2 = \frac{dr^2}{1 - \frac{2M}{r}} + r^2(d\vartheta^2 + \sin^2 \vartheta d\phi^2) - dt^2 \left( 1 - \frac{2M}{r} \right)
\]

a. Use the “advanced time parameter”

\[
p = t + r + 2M \ln |r/2M - 1|
\]

to eliminate \( t \) in the line-element (i.e. introduce Eddington-Finkelstein coordinates) and show that in the new coordinates the singularity at \( R = 2M \) is absent.

b. Draw a space-time diagram considering radial light-rays in the \( t \equiv p - r, r \) plane. Include the world-line of an observer falling into the black hole. Explain why \( r = 2M \) is an event horizon.

c. Determine the smallest possible stable circular orbit of a massive particle. (Hint: Use the Killing vectors of the metric and consider the effective potential \( V_{\text{eff}} \).)

a. Forming the differential,

\[
dp = dt + dr + \left( \frac{r}{2M} - 1 \right)^{-1} dr = dt + \left( 1 - \frac{2M}{r} \right)^{-1} dr,
\]

we can eliminate \( dt \) from the Schwarzschild metric and find

\[
\text{ds}^2 = - \left( 1 - \frac{2M}{r} \right) dp^2 + 2dpdr + r^2 d\Omega.
\]

This metric is regular at \( 2M \) and valid for all \( r > 0 \).

b. For radial light-rays, \( ds = d\phi = d\vartheta = 0 \), it follows

\[
0 = - \left( 1 - \frac{2M}{r} \right) dp^2 + 2dpdr.
\]

There exist three types solutions: i) for \( r = 2M \), light-rays have constant \( r \) and \( p \); ii) light-rays with \( p = \text{const.} \); iii) dividing by \( dp \),

\[
0 = - \left( 1 - \frac{2M}{r} \right) dp + dr
\]

we separate variables and integrate,

\[
p - 2(r + 2M \ln |r/2M - 1|) = \text{const.}
\]
The light-rays of type ii) are ingoing: as $t$ increase, $r$ has to increase to keep $p$ constant. The light-rays of type ii) are ingoing for $r < 2M$ and outgoing for $r > 2M$. Thus for $r < 2M$ both radial light-rays moves towards $r = 0$; all wordlines of observers are inside such light-cones and have to move towards $r = 0$ too. Hence $r = 2M$ is an event horizon.

c. Spherical symmetry allows us to choose $\vartheta = \pi/2$ and $u_\vartheta = 0$. Then we replace in the normalization condition $u \cdot u = -1$ written out for the Schwarzschild metric,

$$-1 = -\left(1 - \frac{2M}{r}\right) \left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1} \left(\frac{dr}{d\tau}\right)^2 + r^2 \left(\frac{d\varphi}{d\tau}\right)^2$$

the velocities $u_t$ and $u_r$ by the conserved quantities

$$e \equiv -\xi \cdot u = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}$$

$$l \equiv \eta \cdot u = r^2 \sin^2 \vartheta \frac{d\varphi}{d\tau}.$$ 

Inserting $e$ and $l$, then reordering gives

$$\frac{e^2 - 1}{2} = \frac{1}{2} \left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}}$$

with

$$V_{\text{eff}} = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3}.$$ 

Circular orbits correspond to $dV_{\text{eff}}/dr = 0$ with

$$r_{1,2} = \frac{l^2}{2M} \left[1 \pm \sqrt{1 - 12M^2/l^2}\right].$$

The stable circular orbit (i.e. at the minimum of $V_{\text{eff}}$) corresponds to the plus sign. The square root becomes negative for $l^2 = 6M$ and thus the “innermost stable circular orbit” is for a Schwarzschild black hole at $r_{\text{ISCO}} = 6M$.

3. Cosmology.

Consider a flat universe dominated by one matter component with E.o.S. $w = P/\rho = \text{const.}$

a. Use that the universe expands adiabatically to find the connection $\rho = \rho(R, w)$ between the density $\rho$, the scale factor $R$ and the state parameter $w$. (4 pts)

b. Find the age $t_0$ of the universe as function of $w$ and the current value of the Hubble parameter, $H_0$. (3 pts)

c. Comment on the value of $t_0$ in the case of a positive cosmological constant, $w = -1$. (2 pts)

d. Find the relative energy loss per time, $E^{-1} dE/dt$, of relativistic particles due to the expansion of the universe for $H_0 = 70\text{km/s/Mpc}$. (1 pt)

a. For adiabatic expansion, the first law of thermodynamics becomes $dU = -PdV$ or

$$d(\rho R^3) = -3PR^2 dR$$
Eliminating $P$ with $P = P(\rho) = w\rho$,
\[
\frac{d\rho}{dR} R^3 + 3\rho R^2 = -3w\rho R^2 .
\]
Separating the variables,
\[
-3(1 + w) \frac{dR}{R} = \frac{d\rho}{\rho} ,
\]
we can integrate and obtain $\rho \propto R^{-3(1+w)}$.

b. For a flat universe, $k = 0$, with one dominating energy component with $w = P/\rho = \text{const.}$ and $\rho = \rho_{\text{cr}} (R/R_0)^{-3(1+w)}$, the Friedmann equation becomes
\[
\dot{R}^2 = \frac{8\pi}{3} G \rho R^2 = H_0^2 R_0^{3+3w} R^{-(1+3w)} ,
\]
where we inserted the definition of $\rho_{\text{cr}} = 3H_0^2/(8\pi G)$. Separating variables we obtain
\[
R_0^{-(3+3w)/2} \int_0^{R_0} dR R^{(1+3w)/2} = H_0 \int_0^{t_0} dt = t_0 H_0
\]
and hence the age of the Universe follows as
\[
t_0 H_0 = \frac{2}{3+3w} .
\]

c. Models with $w > -1$ need a finite time to expand from the initial singularity $R(t = 0) = 0$ to the current value of the scale factor $R_0$, while a Universe with only a $\Lambda$ has no “beginning”, $t_0 H_0 \to \infty$.

d. The connection between the energy $E_0$ today and the energy at redshift $z$ is
\[
E(z) = (1 + z)E_0
\]
and thus $dE = dz E_0$. Differentiating $1 + z = R_0/R(t)$, we obtain with $H = \dot{R}/R$
\[
dz = -\frac{R_0}{R^2} dR = -\frac{R_0}{R^2} \frac{dR}{dt} dt = -(1 + z)H dt .
\]
Combining the two equations, we find $dE = -(1 + z)H dt E_0 = -H dt E$ or
\[
\frac{1}{E} \frac{dE}{dt} = -H(z) = -H_0 (1 + z)^{3/2} .
\]
Numerically, we find for the current epoch
\[
\frac{1}{E} \frac{dE}{dt} \approx \frac{7.1 \times 10^6 \text{cm}}{s} \frac{3.1 \times 10^{24} \text{cm}}{} \approx 5.2 \times 10^{-36} \text{s}^{-1} .
\]
4. Symmetries.
Consider in Minkowski space a complex scalar field $\phi$ with Lagrange density

$$\mathcal{L} = -\frac{1}{2} \partial_a \phi^\dagger \partial^a \phi - \frac{1}{4} \lambda (\phi^\dagger \phi)^2.$$ 

a. Name the symmetries of the Langrangian. (1.5 pts)
b. Derive Noether’s theorem in the form

$$0 = \delta \mathcal{L} = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a - K^\mu \right).$$

c. Derive one conserved current of your choice. (4 pts)

a. space-time symmetries: Translation, Lorentz, scale invariance. internal: global SO(2) / U(1) invariance.

b. We assume that the collection of fields $\phi_a$ has a continuous symmetry group. Thus we can consider an infinitesimal change $\delta \phi_a$ that keeps $\mathcal{L}(\phi_a, \partial_\mu \phi_a)$ invariant,

$$0 = \delta \mathcal{L} = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a \right). \quad (3)$$

Now we exchange $\delta \partial_\mu$ against $\partial_\mu \delta$ in the second term and use then the Lagrange equations, $\delta \mathcal{L}/\delta \phi_a = \partial_\mu (\delta \mathcal{L}/\delta \partial_\mu \phi_a)$, in the first term. Then we can combine the two terms using the Leibniz rule,

$$0 = \delta \mathcal{L} = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \right) \delta \phi_a + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \partial_\mu \delta \phi_a = \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a \right). \quad (4)$$

Hence the invariance of $\mathcal{L}$ under the change $\delta \phi_a$ implies the existence of a conserved current, $\partial_\mu J^\mu = 0$, with

$$J^\mu_1 = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} \delta \phi_a. \quad (5)$$

If the transformation $\delta \phi_a$ leads to change in $\mathcal{L}$ that is a total four-divergence, $\delta \mathcal{L} = \partial_\mu K^\mu$, and boundary terms can be dropped, then the equation of motions are still invariant. The conserved current is changed to

$$J^\mu = \delta \mathcal{L}/\delta \partial_\mu \phi_a \delta \phi_a - K^\mu.$$ 

c. i) Translations: From $\phi_a(x) \rightarrow \phi_a(x - \epsilon) \approx \phi_a(x) - \epsilon^\mu \partial_\mu \phi(x)$ we find the change $\delta \phi_a(x) = -\epsilon^\mu \partial_\mu \phi(x)$. The Lagrange density changes similarly, $\mathcal{L}(x) \rightarrow \mathcal{L}(x - \epsilon)$ or $\delta \mathcal{L}(x) = -\epsilon^\mu \partial_\mu \mathcal{L}(x) = -\partial_\mu (\epsilon^\mu \mathcal{L}(x))$. Thus $K^\mu = -\epsilon^\mu \mathcal{L}(x)$ and inserting both in the Noether current gives

$$J^\mu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_a)} [-\epsilon^\nu \partial_\nu \phi(x)] + \epsilon^\mu \mathcal{L}(x) = \epsilon_\nu T^{\mu \nu}.$$
with $T^{\mu\nu}$ as energy-momentum tensor and four-momentum as Noether charge.

or

ii) Charge conservation: We can work either with complex fields and $U(1)$ phase transformations

$$\phi(x) \rightarrow \phi(x)e^{i\alpha}, \quad \phi^\dagger(x) \rightarrow \phi^\dagger(x)e^{-i\alpha}$$

or real fields (via $\phi = (\phi + i\phi_2)/\sqrt{2}$) and invariance under rotations $SO(2)$. With $\delta\phi = i\alpha\phi$, $\delta\phi^\dagger = -i\alpha\phi^\dagger$, the conserved current is

$$J^\mu = i \left[ \phi^\dagger \partial^\mu \phi - (\partial^\mu \phi^\dagger)\phi \right]$$

**Some formula:** Signature of the metric ($-, +, +, +$).

$$\dddot{x}^c + \Gamma^c_{ab} \dddot{x}^a \dddot{x}^b = 0$$

$$R^a_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce} \Gamma^e_{bd} - \Gamma^a_{ed} \Gamma^e_{bc},$$

$$\frac{e^2 - 1}{2} = \frac{i^2}{2} + V_{\text{eff}}$$

$$H^2 = \frac{8\pi}{3} G\rho - \frac{k}{R^2} + \frac{\Lambda}{3}$$

$$\frac{\dot{R}}{R} = \frac{\Lambda}{3} - \frac{4\pi G}{3} (\rho + 3P)$$

$$1\text{Mpc} = 3.1 \times 10^{24}\text{cm}$$
Fig. 16.10 Schwarzschild solution in advanced Eddington-Finkelstein coordinates.