Problem 1

a) Taking the differentials, we obtain

\[
\begin{align*}
\frac{dt'}{dt} &= \gamma (dt - \frac{v}{c^2} dx) \\
&= \gamma dt \left(1 - \frac{vV_x}{c^2}\right), \\
\frac{dV_x'}{dx} &= \frac{dV_x}{1 - \frac{vV_x}{c^2}} + \frac{V_x - v}{(1 - \frac{vV_x}{c^2})^2} \frac{v}{c^2} dV_x.
\end{align*}
\]

(1)

(2)

Dividing Eq. (2) by Eq. (1), we obtain

\[
\begin{align*}
a'_x &= \frac{1}{\gamma} \frac{a_x}{(1 - \frac{vV_x}{c^2})^2} + \frac{1}{\gamma} \frac{V_x - v}{(1 - \frac{vV_x}{c^2})^3} c^2 \frac{v}{c^2} a_x.
\end{align*}
\]

(3)
If \( S' \) is the instantaneous rest frame, we have \( v = V_x \) and Eq. (3) reduces to

\[
a'_x = \gamma^3 a_x ,
\]

where we have used that \( 1 - \frac{V_x^2}{c^2} = 1 - \frac{V^2}{c^2} = \frac{1}{\gamma^2} \).

b) Since \( a'_x = g \), Eq. (4) can be written as

\[
\frac{dV_x}{dt} = g \left( 1 - \frac{V_x^2}{c^2} \right)^{\frac{3}{2}} .
\]

or

\[
\frac{dV_x}{\left( 1 - \frac{V_x^2}{c^2} \right)^{\frac{3}{2}}} = gdt .
\]

Changing variables, \( V_x = c \sin u \), we obtain

\[
\frac{cdu}{\cos^2 u} = gdt .
\]

Integrating yields

\[
c \tan u = gt + C ,
\]

where \( C \) is an integration constant.

\[
\frac{V_x}{\sqrt{1 - \frac{V_x^2}{c^2}}} = gt + C .
\]

Solving with respect to \( V_x \), this finally yields

\[
V_x(t) = \frac{gt + C}{\sqrt{1 + (gt + C)^2/c^2}} .
\]

\( C = 0 \) since \( V_x(0) = 0 \). Thus

\[
V_x(t) = \frac{gt}{\sqrt{1 + \frac{g^2t^2}{c^2}}} .
\]

The limiting velocity is \( V_{\text{lim}} = \frac{c}{\gamma} \) as seen from Eq. (11).
c) We have
\[
\frac{d\tau}{dt} = \frac{1}{\gamma} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 + \frac{g^2t^2}{c^2}}}
\] (12)

Changing variables \( t = \frac{c}{g} \sinh u \), we can write
\[
d\tau = \frac{c}{g} du.
\] (13)

Integration yields
\[
\tau = \frac{c}{g} \int_0^u du + C = \frac{c}{g} u + C = \frac{c}{g} \sinh^{-1}(\frac{g}{c} t) + K,
\] (14)

where \( K \) is an integration constant. \( K = 0 \) since \( \tau(0) = 0 \). This yields
\[
t(\tau) = \frac{c}{g} \sinh(\frac{g}{c} \tau).
\] (15)

d) Integrating Eq. (11), we find
\[
x(t) = \frac{c^2}{g} \left[ \sqrt{1 + \frac{g^2t^2}{c^2}} - 1 \right],
\] (16)

where we have used that \( x(\tau = 0) = x(t = 0) = 0 \). Substituting Eq. (15) into Eq. (16), we finally obtain
\[
x(\tau) = \frac{c^2}{g} \left[ \cosh(\frac{g}{c} \tau) - 1 \right],
\] (17)

e) Taking the differentials of \( t \) and \( x \) yields
\[
dt = \frac{1}{c} \sinh \left(\frac{gt'}{c}\right) dx' + \left(\frac{c}{g} + \frac{x'}{c}\right) \cosh \left(\frac{gt'}{c}\right) \frac{g}{c} dt',
\] (18)
\[
dx = \cosh \left(\frac{gt'}{c}\right) dx' + \left(\frac{c}{g} + \frac{x'}{c}\right) \sinh \left(\frac{gt'}{c}\right) \frac{g}{c} dt'.
\] (19)
Inserting these expressions into the line element and using $dy = dy'$ and $dz = dz'$, we find

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$

$$= -c^2 dt'^2 \left(1 + \frac{gx'}{c^2}\right)^2 + dx'^2 + dy'^2 + dz'^2,$$

$$= -c^2 dt'^2 \left(1 + \frac{gx'}{c^2}\right)^2 + dx'^2 + dy'^2 + dz'^2,$$

(20)

f) Since the line element is independent of time, the vector $\xi = (1, 0, 0, 0)$ is a Killing vector. The quantity $\xi \cdot p$ is a conserved quantity along a geodesic.

g) A stationary observer with spatial coordinates $(h, 0, 0)$ has four-velocity vector

$$u = \left(\left(1 + \frac{gx'}{c^2}\right)^{-1}, 0, 0, 0\right)$$

$$= \left(1 + \frac{gx'}{c^2}\right)^{-1} \xi.$$  (21)

The energy of a photon with four-momentum $p$ and frequency $\omega$ is $\hbar \omega = -p \cdot u_{\text{obs}}$. This yields

$$\hbar \omega = -\left(1 + \frac{gx'}{c^2}\right)^{-1} \xi \cdot p.$$  (22)

or

$$\hbar \omega \left(1 + \frac{gx'}{c^2}\right) = -\xi \cdot p.$$  (23)

The energy of a photon emitted at $x' = h$ is denoted by $\hbar \omega_h$ and the energy of the same photon absorbed at $x' = h$ is denoted by $\hbar \omega_0$. Eq. (23) then gives

$$\omega_0 = \omega_h \left(1 + \frac{gh}{c^2}\right),$$  (24)

since $\xi \cdot p$ is constant along the photon’s geodesic.

According to the equivalence principle acceleration is equivalent to a gravitational field. The blueshift of the photon is an example of this principle.
Problem 2

(a) Subtracting one-third of the first Friedman equation from the second Friedman equation gives

\[ \ddot{a} = -\frac{4\pi}{3} a \rho_m + \frac{1}{3} a \Lambda. \]  \hspace{1cm} (25)

where we have used that the pressure \( p \) vanishes.

(b) For a time-independent solution, we have \( \dot{a} = \ddot{a} = 0 \). Equation (25), then yields

\[ \rho_m^e = \frac{\Lambda}{4\pi}. \]  \hspace{1cm} (26)

For a static solution the first Friedman equation reduces to

\[ 3 \frac{1}{a_c^2} = 8\pi \rho_m^e + \Lambda, \]  \hspace{1cm} (27)

or

\[ a_c = \frac{1}{\sqrt{\Lambda}}. \]  \hspace{1cm} (28)

c) We write \( a = a_c + \delta a \). Note that \( \dot{a} = \frac{d}{dt} \delta a \) and \( \ddot{a} = \frac{d^2}{dt^2} \delta a \) since \( a_c \) is constant in time. For \( p = 0 \), the second Friedman equation can be rewritten as

\[ 2a \ddot{a} + \dot{a}^2 + 1 = \Lambda a^2. \]  \hspace{1cm} (29)

To first order in the perturbation, Eq. (29) reads

\[ 2a \frac{d^2}{dt^2} \delta a + 1 = \Lambda (a_c^2 + 2a_c \delta a). \]  \hspace{1cm} (30)

Using the result for \( a_c \), we find

\[ \frac{d^2}{dt^2} \delta a = \frac{\Lambda}{\delta a}, \]  \hspace{1cm} (31)

which corresponds to \( B = \Lambda \). This is a second-order differential equation for \( \delta a \), whose solution is

\[ \delta a = A_1 e^{\sqrt{\Lambda} t} + A_2 e^{-\sqrt{\Lambda} t}, \]  \hspace{1cm} (32)

where \( A_1 \) and \( A_2 \) are constants. The perturbation is growing and so the static Einstein universe is unstable. It is the sign of \( B \) that determines the stability of the solution. For \( B < 0 \), the solution for \( \delta a \) would involve trigonometric functions and so the universe would oscillate around the equilibrium solution.