Exam FY3452 Gravitation and Cosmology Spring 2016

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09.00-13.00

Permitted examination support material:
Approved calculator
Rottmann: Matematisk Formelsamling
Rottmann: Matematische Formelsammlung
Barnett & Cronin: Mathematical Formulae
Angell og Lian: Fysiske størrelser og enheter: navn og symboler

The problem set consists of four pages. Read carefully. Good luck! Bonne chance! Viel Glück! Lykke til! In the problems, we use \( c = G = 1 \).

Problem 1

a) Consider two inertial frames \( S \) and \( S' \), where \( S' \) moves along the \( x \)-axis with velocity \( v \). Write down the transformation that expresses \( t' \), \( x' \), \( y' \), and \( z' \) as functions of \( t \), \( x \), \( y \), and \( z \).

A disc of radius \( r \) is rotating counterclockwise with angular speed \( \beta \). Its center is located at the origin in the \( xy \)-plane. A light source on the edge of the disc is emitting radiation at a frequency \( \omega' \) in the rest frame of the source. When the source crosses the \( y \)-axis in the lower half-plane, it emits radiation in the \( y' \) direction, where \( S' \) denotes the instantaneous rest frame.
b) Find the frequency $\omega$ and the components of the wavevector $k$ in the lab frame $S$ in terms of the corresponding quantities $\omega'$ and $k'$ in $S'$.

c) Find the speed $v = \beta r$ such that the angle between the radiation and $x$-axis in $S$ is $\frac{1}{4}\pi$.

**Problem 2**

Consider a two-dimensional space with the line element
\[ ds^2 = dr^2 + f(r)d\phi^2, \quad (1) \]
where $r$ and $\phi$ are coordinates with range $0 \leq r < \infty$ and $0 \leq \phi \leq 2\pi$, and $f(r)$ is a smooth real function.

a) The only nonzero Christoffel symbols are $\Gamma^r_{\phi\phi}$ and $\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r}$. Calculate the nonzero Christoffel symbols.

b) The only nonzero components of the Ricci tensor are $R_{rr}$ and $R_{\phi\phi}$. Calculate the nonzero components of the Ricci tensor.

c) Use this to show that the Ricci scalar $R$ can be written as
\[ R = \frac{1}{2} \frac{[f'(r)]^2}{f^2(r)} - \frac{f''(r)}{f(r)}. \quad (2) \]

d) Finally assume that $f(r)$ is of the form
\[ f(r) = r^n, \quad (3) \]
where $n$ is nonnegative integer. For which values of $n$ is the space flat? For which values is the space Euclidean?

**Problem 3**

In this problem, we are going to study some of the properties of an electrically charge and spherically symmetric black hole. The metric was found by Reissner and Nordstrom and reads
\[ ds^2 = -\left(1 - \frac{2m}{r} + \frac{\varepsilon^2}{r^2}\right)dt^2 + \left(1 - \frac{2m}{r} + \frac{\varepsilon^2}{r^2}\right)^{-1} dr^2 + r^2d\Omega^2, \quad (4) \]
where \( m \) is the mass and \( Q = \varepsilon^2 \) is the electric charge of the black hole.

**a)** \( r = 0 \) is a coordinate singularity (and a physical one as well). Show that the other coordinate singularities are given by

\[
r_{\pm} = m \pm \sqrt{m^2 - \varepsilon^2}.
\]

We can use \( r_{\pm} \) to divide \( r \) into the three different regions according to

\[
I : 0 < r < r_-, \\
II : r_- < r < r_+, \\
III : r_+ < r.
\]

**b)** Using a clever coordinate transformation, the line element can be written in the form

\[
ds^2 = -(1 - f) dt^2 + 2f dt dr + (1 + f) dr^2 + r^2 d\Omega^2,
\]

where \( f = \frac{2m}{r} - \varepsilon^2 \). Show that a family of radial null geodesics are given by

\[
\vec{t} + r = \text{constant}.
\]

Is this family of geodesics incoming or outgoing? Draw the geodesics in an \((\vec{t}, r)\)-diagram.

**c)** Show that there is another family of radial null geodesics given by

\[
\frac{d\vec{t}}{dr} = \frac{1 + f}{1 - f}.
\]

Fig. 1 shows \( 1 - f \) and \( 1 + f \) as functions of \( r \). Use this to sketch the geodesics that are the solutions to Eq. (9) in a \((\vec{t}, r)\)-diagram.

![Figure 1: Plot of \( 1 + f \) and \( 1 - f \) as functions of \( r \). The zeros of \( 1 - f \) are at \( r_{\pm} \).](image-url)
d) Show or explain that $r = r_+$ is an event horizon.

e) Once the particle is in region I, is it bound to fall into the singularity at $r = 0$?

f) We finally specialize to the case where $\varepsilon^2 = \frac{3}{4}m^2$. What are the corresponding values of $r_+$ and $r_-$? Calculate the proper time $\Delta \tau$ it takes for a particle to travel from $r_+$ to $r_-$ starting at rest.

Useful formulas

\[
g_{\alpha\delta} \Gamma^\delta_{\beta\gamma} = \frac{1}{2} \left[ \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} \right], \quad (10)
\]

\[
R_{\alpha\beta} = \partial_\sigma \Gamma^\sigma_{\alpha\beta} - \partial_\beta \Gamma^\gamma_{\alpha\gamma} + \Gamma^\gamma_{\alpha\delta} \Gamma^\delta_{\beta\gamma} - \Gamma^\delta_{\beta\gamma} \Gamma^\gamma_{\alpha\delta}, \quad (11)
\]

\[
R = g^{\alpha\beta} R_{\alpha\beta}. \quad (12)
\]