Exam FY3452 Gravitation and Cosmology fall 2016

Lecturer: Professor Jens O. Andersen
Department of Physics, NTNU
Phone: 73593131 or 46478747 (mob)

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09.00-13.00

Permitted examination support material:
Approved calculator
Rottmann: Matematisk Formelsamling
Rottmann: Matematiscbe Formelsammlung
Barnett & Cronin: Mathematical Formulae
Angell og Lian: Fysiske størrelser og enheter: navn og symboler

The problem set consists of four pages. Read carefully. Good luck! Bonne chance! Viel Glück! Veel succes! Lykke til!

Problem 1

Consider the standard situation where an inertial frame $S'$ moves along the positive $x$-axis with speed $v$ relative to another inertial frame $S$.

a) Show the relation between the acceleration $a'_x$ in $S'$ and $a_x$ in $S$

$$a'_x = \frac{1}{\gamma} \frac{a_x}{(1 - \frac{v}{c^2})^2} + \frac{1}{\gamma} \frac{v}{(1 - \frac{v}{c^2})^3} \frac{va_x}{c^2}$$  \hspace{1cm} (1)

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Show that if $S'$ is the instantaneous rest frame of a particle moving along the $x$-axis in $S$, $a_x'$ reduces to

$$a_x' = \gamma^3 a_x .$$

(2)

b) Assume that the acceleration of the particle moving along the $x$-axis is constant in its instantaneous rest frame and equal to $a_x' = g$. Show that $V_x(t) = \frac{dx}{dt}$ is

$$\frac{dx}{dt} = \frac{gt}{\sqrt{1 + \frac{g^2 t^2}{c^2}}},$$

(3)

if the initial condition is $V_x(0) = 0$. What is the limiting velocity $V_{\text{lim}}$ of $V_x(t)$ as $t \to \infty$?

c) The time $t$ in $S$ can be expressed as a function of the proper time $\tau$ of the particle. Show that

$$t(\tau) = \frac{c}{g} \sinh \left( \frac{g}{c} \tau \right),$$

(4)

if the initial condition is $t(0) = 0$.

d) The position $x$ in $S$ can be expressed as a function of the proper time $\tau$ of the particle. Show that

$$x(\tau) = \frac{c^2}{g} \left[ \cosh \left( \frac{g}{c} \tau \right) - 1 \right],$$

(5)

if the initial condition is $x(0) = 0$.

e) The functions $t(\tau)$ and $x(\tau)$ are the time and position of the origin of $S'$ in $S$. We next consider an arbitrary point in spacetime, whose coordinates in $S'$ are $x'$ and $t' = \tau$. The coordinates of this point in $S$ are given by

$$t = \left[ \frac{c}{g} + \frac{x'}{c} \right] \sinh \left( \frac{g}{c} \tau \right),$$

(6)

$$x = \frac{c^2}{g} \left[ \cosh \left( \frac{g}{c} \tau \right) - 1 \right] + x' \cosh \left( \frac{g}{c} \tau \right).$$

(7)

Show that the metric can be written as

$$ds^2 = -c^2 dt'^2 \left( 1 + \frac{gx'}{c^2} \right)^2 + dx'^2 + dy'^2 + dz'^2.$$

(8)
f) Explain why \( \xi = (1, 0, 0, 0) \) is a Killing vector and find the associated conserved quantity.

g) Calculate the redshift of a photon that is emitted at \( x' = h \) and absorbed at \( x' = 0 \). Explain the result.

**Problem 2**

The two Friedman equations are given by

\[ \frac{3\dot{a}^2 + k}{a^2} = 8\pi p + \Lambda, \quad (9) \]

\[ \frac{2\ddot{a}a + \dot{a}^2 + k}{a^2} = -8\pi p + \Lambda, \quad (10) \]

where \( a(t) \) is the scale factor, \( k = 0, \pm 1 \) is the spatial curvature, \( p \) is the energy density of matter and radiation, \( \rho \) is the pressure, and \( \Lambda > 0 \) is the cosmological constant.

a) In Einstein’s static model for the universe, there is no radiation present \( (p = \rho_m) \) and the pressure vanishes. Moreover the spatial curvature is positive, \( k = +1 \). Show that

\[ \ddot{a} = -\frac{4\pi}{3} a\rho_m \frac{1}{3} a\Lambda. \quad (11) \]

b) For a given value of \( \Lambda \), there is a critical value of \( \rho_m, \rho_m^c \), such that \( a \) is time independent. Find the value of \( \rho_m^c \) in terms of \( \Lambda \). Find the corresponding value of \( a = a_c \) in terms of \( \Lambda \).

c) We will next study the stability of the static universe. We consider a small perturbation \( \delta \rho_m \) of the density around \( \rho_m^c \) and write \( \rho_m = \rho_m^c + \delta \rho_m \). We can then write \( a = a + \delta a \), where \( \delta a \) is the corresponding change in the scale factor. \( \delta \rho_m \) and \( \delta a \) are time dependent. Use the Friedman equations to show that \( \delta a \) satisfies the second-order differential equation

\[ \frac{d^2\delta a}{dt^2} = B\delta a, \quad (12) \]

where \( B \) is a constant. Calculate \( B \). Use this result to determine whether Einstein’s static universe is stable or unstable. (Help: Even if you cannot find \( B \), you can still say something about the stability).
Useful formulas

\[ x' = \gamma (x - vt), \]  \hspace{1cm} (13)

\[ t' = \gamma \left( t - \frac{v}{c^2} x \right), \]  \hspace{1cm} (14)

\[ V_x' = \frac{V_x - v}{1 - \frac{v}{c^2}}. \]  \hspace{1cm} (15)